Solutions for HW 6

Exercise 1.2.18: Solution:

(i.) Without los of generality we can assume $E \subset A \subset A'$ (otherwise replace A by $A \cap A'$). Then $m(A') - m(A) = m(A' \setminus A)$. Thus we have to show

$$m^*(A \setminus E) + m(A' \setminus A) = m^*(A' \setminus E).$$

Let G be a measurable set with $A \supset G \supset A \setminus E$ such that $m(G) = m^*(A \setminus E)$. Similarly, let G_1 be a measurable set with $A' \supset G_1 \supset A' \setminus E$ such that $m(G_1) = m^*(A' \setminus E)$. Let $F = (G_1 \setminus (A' \setminus A)) \cap G$. Then $F \supset A \setminus E$ and $m(F) = m^*(A \setminus E)$. Also $G_1 \supset F \cup (A' \setminus A) \supset A' \setminus E$, so $m(F \cup (A' \setminus A)) = m^*(A' \setminus E)$. This shows $m^*(A' \setminus E) = m(F) + m(A' \setminus A) = m^*(A \setminus E) + m(A' \setminus A)$ and the proof of (i) is complete. An alternate way of showing (i) is to show that $m_*(E) = \sup\{m(K) : K \subset E, K \text{ compact}\}$. To see this, observe

$$\begin{aligned} m^*(A \setminus E) &= \inf\{m(U) : A \setminus E \subset U, U \text{ open}\} \\ &= \inf\{m(U \cap A) : A \setminus E \subset U, U \text{ open}\} \\ &= \inf\{m(U \cap A) : E^c \subset U, U \text{ open}\} \\ &= \inf\{m(A \setminus K) : K \subset E, K \text{ compact}\} \\ &= m(A) - \sup\{m(K) : K \subset E, K \text{ compact}\} \end{aligned}$$

(ii.) Let first $m_*(E) = m^*(E)$. Then find a measurable G with $A \supset G \supset E$ and $m(G) = m^*(E)$. Similarly find a measurable set H with $A \supset H \supset A \setminus E$ and $m(H) = m^*(A \setminus E)$. Let $F = A \setminus H$. Then $F \subset E \subset G \subset A$ and $m(G \setminus F) = m(G) - m(F) = m^*(E) - (m(A) - m(H)) = m^{*(E) - m_*(E} = 0$. It now follows easily that E is measurable. If we use the other identity for $m_*(E)$, then for all $\epsilon > 0$ we can find an open set $U \supset E$ with $m(U) < m^*(E) + \frac{\epsilon}{2}$ and a compact set $K \subset E$ with $m(K) > m^*(E) - \frac{\epsilon}{2}$. Now $m(U \setminus K) = m(U) - m(K) < \epsilon$ and the measurability of E follows. If E is measurable, then $m^*(A \setminus E) = m(A) - m(E) = m(A) - m^*(E)$, so $m_*(E) = m^*(E)$.

Exercise 1.3.8: Solution: (vi) There exist sequences (f_n) and (g_n) of bounded simple functions such that $f_n(x) \to f(x)$ and $g_n(x) \to g(x)$ for all $x \in \mathbb{R}^d$. Then $f_n(x) + g_n(x)$ and $f_n g_n$ are simple functions and $f_n(x) + g_n(x) \to f(x) + g(x)$ as well as $f_n(x) g_n(x) \to f(x) g(x)$ for all x. Hence $f + g$ and fg are measurable.

Exercise 1.3.15: Solution: From Hw 4 we know that if E measurable, then $E + x$ measurable and $m(E) = m(E + x)$. This implies immediately that if g is an signed simple function, then $g(\cdot + y)$ is an unsigned simple function and $\int g(x)\, dx = \int g(x + y)\, dx$. Now, if $0 \le g \le f$, g simple than $0 \le g(\cdot + y) \le f(\cdot + y)$ and if $0 \le g \le f(\cdot + y)$, then $0 \le g(\cdot - y) \le f$. By taking supreme we get that $\int f(x)\, dx = \int f(x + y)\, dx$.

Problem 1: Solution: Assume $\limsup_{x \to \infty} |f(x)| = \epsilon > 0$. Then by uniform continuity there exists $\delta > 0$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \frac{\epsilon}{4}$. Then there exist $x_n \uparrow$, $x_n \to \infty$ such that $x_{n+1} - x_n > 2\delta$ and $|f(x_n)| > \frac{\epsilon}{2}$. This implies that $|f(y)| > \frac{\epsilon}{4}$ for all $y \in (x_n - \delta, x_n + \delta)$. Hence $\int_{x_n - \delta}^{x_n + \delta} |f|\, dx \ge \frac{\delta \epsilon}{2}$ for all n. But $x_{n+1} - x_n > 2\delta$ implies $x_n + \delta < x_{n+1} - \delta$ so that the intervals $(x_n - \delta, x_n + \delta)$ are disjoint. Hence $\int |f|\, dx \ge \sum_n \int_{x_n - \delta}^{x_n + \delta} |f|\, dx = \infty$. This contradiction with the integrability of f shows that $\lim_{x \to \infty} |f(x)| = 0$.

To get a counter example define f piecewise. Let $\epsilon_n = \frac{1}{2^n}$. Define first f on $[2, \infty)$ as follows. For $n \ge 2$ let $f(n) = f(n + \frac{1}{n^3}) = \epsilon_n$, $f(n + \frac{1}{2n^3}) = n$, and f linear and continuous on

$[n, n + \frac{1}{2n^3}]$, $[n + \frac{1}{2n^3}, n + \frac{1}{n^3}]$, and $[n + \frac{1}{n^3}, n + 1]$. Then f positive and continuous on $[2, \infty)$ and $\int_n^{n+1} f\,dx \leq \frac{1}{2}\frac{1}{n^2} + \epsilon_n$ implies that $\int_2^\infty f\,dx < \infty$. Moreover $\limsup_{x\to\infty} f(x) = \infty$. To get a function on $\mathbb{R}$ extend the above f by defining $f(x) = \epsilon_2 e^{x-2}$ on $(-\infty, 2]$.