Solutions homework 5.

Page 128 Problem 3: Using the substitution $y = -x$ we get

$$\frac{g(y) - g(-c)}{y - (-c)} = \frac{f(-y) - f(c)}{y - (-c)} = -\frac{f(x) - f(c)}{x - c}.$$

Now letting $y \to -c^-$ is the same as letting $x \to c^+$, from which the problem follows.

Page 128 Problem 4. As $g(x) = 0$ for $x \geq 0$ we clearly have $g_r'(0) = 0$. On the other hand $(g \circ f)(x) = x \sin(\frac{1}{x})$ for $x > 0$ and $(g \circ f))(0) = 0$. Hence

$$\frac{(g \circ f)(x) - (g \circ f)(0)}{x - 0} = \sin(\frac{1}{x}),$$

for $x > 0$. It is not difficult to se that $\lim_{x \to 0^+} \sin(\frac{1}{x})$ doesn't exist (consider e.g. what happens if $x_n = \frac{1}{\pi n}$ and $x_n = \frac{1}{\frac{\pi}{2} + 2\pi n}$).

Page 129 Problem 2. True, if f has a maximum at an interior pt c, then for any neighborhood $V \subset S$ of c we have $f(x) \leq f(c)$ on V.

Page 132 Problem 2.

a. If there exist $a < b$ in I such that $f(a) = f(b)$, then by Rolle's theorem there exists $a < c < b$ such that $f'(c) = 0$, which contradicts the assumption that $f'(x) \neq 0$ for all x.

b. Assume $f(J)$ is not an open interval. Then there exists $c \in f(J)$ which is an endpoint of $f(J)$ and thus not interior to $f(J)$. Now $c = f(a)$ for some $a \in I$. As I is open there exist $\epsilon > 0$ that $(a - \epsilon, a + \epsilon) \subset I$. By strict monotonicity of f we have either $c = f(a) \in (f(a - \epsilon), f(a + \epsilon)) \subset f(J)$ or $c = f(a) \in (f(a + \epsilon), f(a - \epsilon)) \subset f(J)$, which shows that c is interior to $f(J)$. Contradiction.

c. If $(a, b) \subset I$, then there exists an open interval $(c, d) \subset J$ such that $f((a, b)) = (c, d)$ (by essentially the same argument that showed that $f(I) = J$). Hence $g^{-1}((a, b))$ is open for all open intervals in I. This shows that g is continuous. Now

$$\frac{g(y) - g(c)}{y - c} = \frac{g(y) - g(c)}{(f \circ g)(y) - (f \circ g)(c)} = \frac{g(y) - g(c)}{(f(g(y)) - (f(g(c))}.$$

If $y \to c$, then $g(y) \to g(c)$. Hence by the differentiability of f we get that the right hand side has limit $1/f'(g(c))$ as $y \to c$. It follows that g is differentiable at c and $g'(c) = 1/f'(g(c))$.

Page 132 Problem 4. Let $L = \lim_{x \to c} f'(x)$. Then if $x_n \to c$with all $x_n \neq c$ in (a, b) we have by the Mean Value Theorem that

$$\frac{f(x_n) - f(c)}{x - c} = f'(y_n),$$

where y_n is strictly between x_n and c. As $x_n \to c$ we have therefore that also $y_n \to c$, so $f'(y_n) \to L$ as $x \to c$. Hence $f'(c)$ exists and equals L.

Page 132 Problem 5. Let $F(x) = [f(b) - f(a)]g(x) - [g(b) - g(a)]f(x)$. Then F is continuous on $[a, b]$ and differentiable on (a, b). Moreover by plugging in we see that $F(a) = F(b)$. It follows from Rolle's theorem that there exist $c \in (a, b)$ such that $F'(c) = 0$, i.e., $[f(b) - f(a)]g'(c) - [g(b) - g(a)]f'(c) = 0$. The result follows now if we can divide by $[g(b) - g(a)]$, i.e., it follows if $[g(b) - g(a)] \neq 0$. This follows however from Rolle's theorem as $g'(x) \neq 0$ for all $x \in (a, b)$.

Page 132 Problem 7. As the hint states we can assume that $f, g : (a, b) \to \mathbb{R}$ continuous and differentiable on $(a, b) \setminus \{c\}$, where $a < c < b$. Furthermore $f(x) \to 0$, $g(x) \to 0$ (so that

$f(c) = g(c) = 0$) and $f'(x)/g'(x) \to L$ as $x \to c$. Moreover $g'(x) \neq 0$ on $(a,b) \setminus \{c\}$. Then by Problem 5 we have that if $x_n \to c$ there exist t_n strictly between x_n and c such that

$$\frac{f(x_n)}{g(x_n)} = \frac{f(x_n) - f(c)}{g(x_n) - g(c)} = \frac{f'(t_n)}{g'(t_n)}.$$

As $x_n \to c$ we also have $t_n \to c$ and thus $f'(t_n)/g'(t_n) \to L$ and the conclusion follows.

Page 132 Problem 8. False. By the Mean Value Theorem $f(3) - f(1) = f'(c)(3-1) = 2f'(c)$. By assumption $f'(c) = f(c)^2 + 4 \geq 4$, so $f(3) - f(1) \geq 2 \cdot 4 = 8 > 5$.