Solutions homework 5.

Problem 1 = Problem 28-2. Solution: Yes the function is differentiable at 0. To see this, note that for $x > 0$ we have $0 \le \frac{f(x)-f(0)}{x-0} \le \frac{x^2}{x} = x$, so that $\lim_{x\to 0+} \frac{f(x)-f(0)}{x-0} = 0$. Similarly for $x \to 0-$ we have that this limit is 0, so $f'(0) = 0$.

Problem 2 = Problem 33-3. Solution: Let $s_n = f_1 + \cdots + f_n$. By assumption $\{s_n\}$ converges pointwise to the continuous f on the compact set A. However we have $s_1 \le s_2 \le \cdots s_n \le \cdots$, as $f_n \ge 0$. Hence the sequence $\{s_n\}$ converges by Dini's Theorem uniformly to f.

Problem 3 = Problem 33-6. Solution: At $x = 0$ we have $f_n(0) = 1$ for all $n \ge 1$, so the limit $\lim_{n\to\infty} f_n(0) = 1$. If $0 < x \le 1$, then we can find N such that $\frac{1}{N} < x$. This implies that $f_n(x) = 0$ for all $n \ge N$ and thus $\lim_{n\to\infty} f_n(x) = 0$ for all $0 < x \le 1$. Hence the sequence converges pointwise on $[0, 1]$. As the limit function is discontinuous at 0 the convergence can't be uniform (one can see this also from $\|f_n\| = 1$ for all n).

Problem 4 = Problem 33-11. Solution: To apply Weierstrass' M-theorem we need to estimate $f_n(x) = x^n e^{-nx}$. Using the product rule we find that $f_n'(1) = 0$ and that $f_n'(x) > 0$ for $0 < x < 1$ and $f_n'(x) < 0$ for $x > 1$. Hence f_n has a maximum at $x = 1$, i.e., $0 \le f_n(x) \le f_n(1) = e^{-n} = (\frac{1}{e})^n = M_n$. As $0 < \frac{1}{e} < 1$ we have $\sum M_n < \infty$ and thus by Weierstrass' theorem the series converges uniformly.

Problem 5 = Problem 33-12. Solution: Note first that

$$\frac{nx^2}{n^3 + x^3} \le \frac{x^2}{n^2 + \frac{x^3}{n}} \le \frac{x^2}{n^2}.$$

Now $0 \le \frac{x^2}{n^2} \le \frac{4}{n^2}$ and $\sum \frac{1}{n^2} < \infty$ implies that the series converges uniformly on $[0, 2]$. Therefore the sum of the series is continuous at $x = 1$ and the result follows.