Solutions homework 5.

(1) **Problem 8-9.** Let $\epsilon > 0$ be a (rational) number. Then there exists $n_0 \in \mathbb{N}$ such that $d(a_k, a_l) < \epsilon$ for all $k, l \geq n_0$. Now the inequality $|\,|a_k| - |a_l|\,| \leq |a_k - a_l|$ implies that $d(|a_k|, |a_l|) \leq d(a_k, a_l) < \epsilon$ for all $k, l \geq n_0$. This shows that $\{|a_k|\}$ is a Cauchy sequence. The converse does not hold: take e.g. $a_k = (-1)^k$. Then $\{|a_k|\} = (1, 1, 1, \cdots)$, which is Cauchy, but $\{a_k\}$ is not Cauchy since $d(a_k, a_{k+1}) = 1$ for all k.

(2) **Problem 8-10.** As $\{\alpha_k\}$ is Cauchy it has a limit β in $\mathbb{R}$. From the inequality $|\,|\beta| - |\alpha_k|\,| \leq |\beta - \alpha_k|$ it follows that $|\beta| = \lim |\alpha_k| = \alpha$. Now $|\beta| = \beta$ or $|\beta| = -\beta$, so the result follows. Note the condition $\alpha \neq 0$ is not needed as long as we interpret the "or" as an inclusive or.

(3) **Problem 8-11.** Assume $\lim a_k = \lim b_k = 0$. Then for given $\epsilon > 0$ there exist n_1, n_2 such that $|a_k| = |a_k - 0| < \frac{\epsilon}{2}$ for $k \geq n_0$ and $|b_k| = |b_k - 0| < \frac{\epsilon}{2}$ for $k \geq n_1$. Let $k_0 = \max\{n_0, n_1\}$. Then $|a_k + b_k| \leq |a_k| + |b_k| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$ for all $k \geq k_0$. This shows that $\lim a_k + b_k = 0$. For the product for given $\epsilon > 0$ there exist n_1, n_2 such that $|a_k| = |a_k - 0| < \epsilon$ for $k \geq n_0$ and $|b_k| = |b_k - 0| < 1$ for $k \geq n_1$. Let $k_0 = \max\{n_0, n_1\}$. Then $|a_k b_k| = |a_k||b_k| < \epsilon$ for all $k \geq k_0$. This shows that $\lim a_k b_k = 0$. Note for this part we really only need that one of the sequences is a null sequence and that the other one is bounded.