Solutions for HW 5

Exercise 1.2.9: Solution: Since I_n is a finite union of closed intervals, it is closed. Hence C as an intersection of closed sets is also closed. As C is obviously bounded it follows that C is compact. To show that C is uncountable, assume $C = \{x_n : n = 1, 2, \cdots\}$. Then x_1 belongs to exactly on of the two closed intervals of I_1. Let J_1 be the closed interval of I_1, which does not contain x_1. Now J_1 is the union of two closed disjoint intervals of I_2. Let $J_2 \subset J_1 \cap I_2$ be the closed subinterval which does not contain x_2. Continuing like this we get closed nested intervals $J_n \subset J_{n-1} \subset \cdots \subset J_1$ such that $J_n \subset I_n$ and $x_k \notin J_n$ for $1 \leq k \leq n$. By the Nested Interval Theorem we have $\cap_{n=1}^{\infty} J_n = \{x\}$. Then $x \in C$, but $x \neq x_n$ for all n by construction. This is a contradiction. Hence C is uncountable. To prove $m(C) = 0$, note $m(C) \leq m(I_n) = \frac{2^n}{3^n}$.

Exercise 1.2.26: Solution: We observe first that if E is the non-measurable set constructed in class (or text) and $F \subset E$ is measurable, then $m(F) = 0$. Let $\{q_k\} = \mathbb{Q} \cap [-1, 1]$ and $E_k = q_k + E$ as in the book or notes. Let $F_k = q_k + F$. Then F_k measurable and $m(F) = m(F_k)$. Now $F_k \subset E_k$ implies that $\{F_k\}$ is a disjoint collection and $\cup_k F_k \subset [-1, 2]$. This implies that $\sum_{k=1}^{\infty} m(F) = m(\cup F_k) \leq m([-1, 2]) = 3$. Hence $m(F) = 0$. We now claim $m^*([0, 1] \setminus E) = 1$, from which the claim of the exercise follows. Assume therefore $m^*([0, 1] \setminus E) < 1$. Then there exist a (relatively) open set $U \subset [0, 1]$ such the $m(U) < 1$ and $[0, 1] \setminus E \subset U$. Now $F = [0, 1] \setminus U$ is a closed subset E of positive measure, which is contradiction.

Problem 1: Solution:

a. See solution of above problem.

b. Let $A_n = [n, n+1] \cap A$ for $n \in \mathbb{Z}$. Then $A = \cup_{n=-\infty}^{\infty} A_n$ implies that there exists $n \in \mathbb{Z}$ such that $m^*(A_n) > 0$. Let $B = -n + A_n$. Then $m^*(B) > 0$ and $B \subset [0, 1]$. If $F \subset B$ is non-measurable, then $n + B \subset A_n \subset A$ non-measurable. Since $\cup_k E_k \supset [0, 1]$ we have $B = \cup_k B \cap E_k$. If each $B \cap E_k$ measurable, then $m(B \cap E_k) = 0$ for all k and thus $m^*(B) = 0$, which contradicts our assumption. Hence there exists a k such that $F = B \cap E_k$ is non-measurable.

Problem 2: Solution:

(1) Let $A_n = A \cap \overline{B(0; n)}$ and $B_n = B \cap \overline{B(0; n)}$. Then A_n, B_n closed and bounded, so A_n, B_n are compact. Moreover $A + B = \cup_n A_n + B_n$. To prove $A + B$ measurable it suffices therefore to show $A_n + B_n$ is measurable. This is true however, since in fact $A_n + B_n$ is compact. To see this, let $\{a_m + b_m\}$ be a sequence in $A_n + B_n$. Then $\{a_m\}$ has a convergent subsequence $\{a_{m_k}\}$ such that $a_{m_k} \to a \in A_n$. Now $\{b_{m_k}\}$ has a further subsequence $\{b_{m_{k_l}}\}$ such that $b_{m_{k_l}} \to b \in B_n$. Now $\{a_{m_{k_l}} + b_{m_{k_l}}\}$ is a subsequence of $\{a_m + b_m\}$ which converges to $a + b \in A_n + B_n$. Hence $A_n + B_n$ is compact, and thus measurable.

(2) It suffices to show $[0, 1] \subset \mathcal{C} + \frac{1}{2}\mathcal{C}$. To see this, let $x \in [0, 1]$. Then x has a ternary expansion $x = \sum_{n=1}^{\infty} \frac{a_n}{3^n}$, where $a_n \in \{0, 1, 2\}$. Let $b_n = a_n$ in case $a_n = 2$ and $b_n = 0$ otherwise and $c_n = a_n$ in case $a_n = 1$ and $c_n = 0$ otherwise. Define $y = \sum_{n=1}^{\infty} \frac{b_n}{3^n}$ and $z = \sum_{n=1}^{\infty} \frac{c_n}{3^n}$. Then $x = y + z$ and $y \in \mathcal{C}$ and $z \in \frac{1}{2}\mathcal{C}$.

Problem 3: Solution:

a. If $\tilde{I}_k$ denotes the closed set remaining at the k-th stage, then $m(\tilde{I}_k) = 1 - \sum_{j=1}^{k} \frac{\epsilon}{2^j}$. Now $m(\tilde{\mathcal{C}}) = \lim_{k\to\infty} m(\tilde{I}_k) = 1 - \epsilon$.

b. One can see by induction that the length of each closed interval in $\tilde{I}_k$ has length less than $\frac{1}{2^k}$. Let $x \in \tilde{\mathcal{C}}$. Then, if we pick x_k as the center of the nearest open interval to x removed at stage k, then $|x - x_k| < \frac{1}{2^{k-1}} \to 0$.

c. If we pick one of the endpoints $y_k \neq x$ of the removed open interval at stage k, then $y_k \in \tilde{\mathcal{C}}$ and $|x - y_k| \to 0$. Hence $\tilde{\mathcal{C}}$ is perfect and by the previous part it has no interior points.

d. **First Solution:** Assign to each $x \in \tilde{\mathcal{C}}$ an infinite sequence of zeros and ones as follows: If at stage k the point x lies to the left of the removed interval assign it a zero and a one if it lies to the right. Conversely we can associate to each sequence of zeros and ones a unique sequence of nested closed intervals of length less than $\frac{1}{2^k}$, so that there is a unique $x \in \tilde{\mathcal{C}}$ which is in the intersection of these intervals. We have in this way a one to one correspondence between the elements in $\tilde{\mathcal{C}}$ and the set of all sequences of zeros and ones, which is uncountable. Hence $\tilde{\mathcal{C}}$ is uncountable.

Second Solution: Since $m(\tilde{\mathcal{C}}) > 0$ we have that $\tilde{\mathcal{C}}$ is uncountable.