Solutions homework 4.

Page 108 Problem 4: Let (x_n) be a Cauchy sequence and let $\epsilon > 0$. Then there exists $\delta > 0$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon$. For this δ there exist N such that $|x_n - x_m| < \delta$ for all $n, m \geq N$. Hence $|f(x_n) - f(x_m)| < \epsilon$ for all $n, m \geq N$, i.e., $(f(x_n))$ is a Cauchy sequence.

Page 108 Problem 5.

a. If f and g are uniformly continuous, then so is $f + g$. Proof: Let $\epsilon > 0$ then there exists $\delta_1 > 0$ such that $|x - y| < \delta_1$ implies $|f(x) - f(y)| < \frac{\epsilon}{2}$. Similarly there exists $\delta_2 > 0$ such that $|x - y| < \delta_2$ implies $|g(x) - g(y)| < \frac{\epsilon}{2}$. let $\delta = \min\{\delta_1, \delta_2\}$. Then $\delta > 0$ and $|x - y| < \delta$ implies $|f(x) + g(x) - f(y) - g(y)| < \epsilon$. For the proof that $|f|$ is uniformly continuous, let $\delta > 0$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon$. Then the inequality $||f(x)| - f(y)|| \leq |f(x) - f(y)|$ implies that $|x-y| < \delta$ implies $||f(x|) - |f(y)|| < \epsilon$, i.e. $|f|$ is uniformly continuous. Now $\sup(f, g)$ and $\inf(f, g)$ are uniformly continuous by the formulas given in class. The product of two uniformly continuous does not need to be uniformly continuous, e.g., take $S = \mathbb{R}$ and $f(x) = g(x) = x$. Then f, g are uniformly continuous, but fg is not.

b. Let $\epsilon > 0$. Then there exists $\delta > 0$ such that$|x - y| < \delta$ implies $|g(x) - g(y)| < \epsilon$. Now $\delta_1 > 0$ such that $|x - y| < \delta_1$ implies $|f(x) - f(y)| < \delta$. Now combining these two we see that $|x - y| < \delta_1$ implies $|g(f(x)) - g(f(y))| < \epsilon$.

Page 108 Problem 6. Assume f is unbounded. Then there exist $x_n \in S$ such that $f(x_n)| \geq n$. Now S bounded implies that (x_n) has a convergent subsequence (x_{n_k}). Now (x_{n_k}) is Cauchy implies by problem 4 that $(f(x_{n_k}))$ is Cauchy. Now Cauchy sequences are bounded, so $(f(x_{n_k}))$ is bounded, which contradicts $|f(x_{n_k})| \geq n_k \to \infty$.

Page 123 Problem 4. From the inequality $|f(x)| \leq |x|$ it follows immediately that f is continuous at 0. If $0 < x_n \in \mathbb{Q}$ and $x_n \to 0$, then $\frac{f(x_n)-f(0)}{x_n-0} = 1$ for all n, while $\frac{f(x_n)-f(0)}{x_n-0} = 1$ for all n, when x_n is irrational. Hence f is not right differentiable at 0. The argument is the same for left differentiability.

Page 123 Problem 5.

a. Straightforward algebra shows that

$$\frac{f(b_n) - f(a_n)}{b_n - a_n} - f'(c) = [g(b_n) - f'(c)]\alpha_n + [g(a_n) - f'(c)]\beta_n.$$

Now $[g(b_n) - f'(c)]\alpha_n \to 0$ as $n \to \infty$, since $[g(b_n) - f'(c)] \to 0$ and α_n is bounded. Similarly the other term on the right tends to 0. Hence the term on the left has limit zero.

b. Take f as in the hint. Then $0 \leq f(x) \leq x^2$ implies that f has derivative zero at 0. Taking a_n as in the hint, it is clear that the conclusion of (i) fails.

Extra problem 1. Let (x_n) and (y_n) be such that $x_n - y_n \to 0$. Then

$$f(x_n)g(x_n) - f(y_n)g(y_n) = [f(x_n) - f(y_n)]g(x_n) - f(y_n)[g(x_n) - g(y_n)] \to 0,$$

as the terms on the right go to zero or are bounded.

Extra Problem 2. Since f is bounded on $[0, c]$, it is by periodicity bounded on $\mathbb{R}$. The restriction of f to $[0, 2c]$ is continuous, and thus uniformly continuous on $[0, 2c]$. Let now $\epsilon > 0$. Then there exists $0 < \delta < c$ such that $|x - y| < \delta$, $x, y \in [0, 2c]$ implies that $|f(x) - f(y)| < \epsilon$. Let now $x < y \in \mathbb{R}$ such that $y - x < \delta$. Then there exists an integer n such that $\leq x - n\delta < \delta < c$. Then $y - x < \delta < c$ implies that $y - n\delta \in [0, 2c]$. Now $|(y - n\delta) - (x - n\delta)| < \delta$ implies that $|f(x - n\delta) - f(y - n\delta)| < \epsilon$. It follows now from $f(x) = f(x - n\delta)$ and $f(y) = f(y - n\delta)$ that $|f(x) - f(y)| < \epsilon$.