Solutions homework 4.

Problem 1 = Problem 30-1. Solution: Note that $\frac{d(f(x),f(y))}{d(x,y)} \leq cd(x,y)^{\alpha-1}$ implies by taking the limit as $y \to x$ that $|f'(x)| = 0$ for all x, and thus f is constant.

Problem 2 = Problem 30-8 Solution:

(1) Note $x(\sqrt{x^2+1}-x) = \frac{x(x^2+1-x^2)}{\sqrt{x^2+1}+x} = \frac{x}{\sqrt{x^2+1}+x} \to \frac{1}{2}$ as $x \to \infty$ by dividing first both numerator and denominator by x.

(2) Observe $x^x = e^{x \ln x}$, so by the next limit we have $\lim_{x\to 0+} x^x = e^0 = 1$.

(3) By L'Hopital's rule $\lim_{x\to 0+} \frac{\ln x}{\frac{1}{x}} = \lim_{x\to 0+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0+} -x = 0$.

(4) Put $t = \frac{1}{x}$. Then $\lim_{x\to\infty} x^{\frac{1}{x}} = \lim_{t\to 0+} \frac{1}{t^t} = 1$. Note you can also write $x^{\frac{1}{x}} = e^{\frac{\ln x}{x}}$ and use L'Hospital for the limit $\lim_{x\to\infty} \frac{\ln x}{x} = 0$.

(5) By L'Hospital's rule $\lim_{x\to\infty} \frac{\ln(1+e^x)}{x} = \lim_{x\to\infty} \frac{e^x}{1+e^x} = 1$.

Problem 3 = Problem 30-9. Solution: f differentiable at x implies that we can write $f(y) = f(x) + (y-x)f'(x) + (y-x)\eta(y)$, where $\eta(y) \to 0$ as $y \to x$. This implies that

$$\frac{f(x+h_n) - f(x-k_n)}{h_n+k_n} = \frac{h_n f'(x) + h_n\eta(x+h_n) + k_n f'(x) + k_n\eta(x-k_n)}{h_n+k_n}$$
$$= f'(x) + \frac{h_n}{h_n+k_n}\eta(x+h_n) + \frac{k_n}{h_n+k_n}\eta(x-k_n) \to f'(x)$$

as $n \to \infty$, as $0 \leq \frac{h_n}{h_n+k_n} \leq 1$ and $\eta(x+h_n) \to 0$ and similarly for the other term.

Problem 4 Solution: For $|x| < 1$ we have that $x^{2n} \to 0$ as $n \to \infty$. Hence $\lim_{n\to\infty} f_n(x) = 0$ for $|x| < 1$. If $x = \pm 1$, then $f_n(x) = \frac{1}{2}$ for all $n \geq 1$, so $\lim_{n\to\infty} f_n(x) = \frac{1}{2}$ for $x = \pm 1$. For $|x| > 1$ we have that $f_n(x) = \frac{1}{\frac{1}{x^{2n}}+1} \to 1$ as $n \to \infty$. Obviously f is discontinuous at $x = \pm 1$.

Problem 5. Solution: For $x = 0$ we have $f_n(0) = 0$ for all n, so $\lim_{n\to\infty} f_n(0) = 0$. For $x \neq 0$ put $t_n = nx^2$. Then $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{1}{x}\frac{t_n}{e^{t_n}}$. Now by L'Hospital's rule $\lim_{y\to\infty} \frac{y}{e^y} = \lim_{y\to\infty} \frac{1}{e^y} = 0$. Hence $\lim_{n\to\infty} f(x) = 0$. Hence for all $x \in \mathbb{R}$ we have $f_n(x) \to 0$. To see that f_n does not converge uniformly to 0, note that $\|f_n\| \geq |f_n(x_n)| = \sqrt{n}e^{-1} \to \infty$ instead of $\|f_n\| \to 0$.