Solutions homework 4.

(1) **Problem 6-9.**

a. $d(a \cdot b, a \cdot c) = |ab - ac| = |a(b - c)| = |a| \cdot |b - c| = |a| \cdot d(b, c)$.

b. $d(a+b, c+d) = |a+b-c-d| = |a-c+b-d| \leq |a-c|+|b-d| = d(a, c)+d(b, d)$.

(2) **Problem 6-10.** Let $\epsilon > 0$. Then there exists $n_0 \in \mathbb{N}$ such that $d(a_m, a_l) < \frac{\epsilon}{2}$ for all $m, l \geq n_0$. Also there exists $k_0 \in \mathbb{N}$ such that $d(a_{2k}, a) < \frac{\epsilon}{2}$ for all $k \geq k_0$. Let $n_1 = \max\{n_0, 2k_0\}$. We claim $d(a_n, a) < \epsilon$ for all $n \geq n_1$. Let $n \geq n_1$. If n is even, then $n = 2k$ for some $k \geq k_0$, so $d(a_n, a) = d(a_{2k}, a) < \frac{\epsilon}{2} < \epsilon$. If n is odd, then $n + 1$ even, so $d(a_{n+1}, a) < \frac{\epsilon}{2}$. Moreover $d(a_n, a_{n+1}) < \frac{\epsilon}{2}$. Hence $d(a, a_n) \leq d(a, a_{n+1}) + d(a_{n+1}, a_n) < \epsilon$.

(3) **Problem 8-1.** If we identify the equivalence of the rational number with the number itself (which means we work with the constant sequence as the representative of it's equivalence class), then the proof by contradiction is as follows : if $\alpha + \beta = \gamma$ is rational, then $\beta = \gamma - \alpha$ is rational, which is a contradiction. Similarly, if $\alpha \cdot \beta = \gamma$ is rational, then $\beta = \frac{\gamma}{\alpha}$ is rational.

(4) **Problem 8-2.** Let $\alpha = \{a_n\}$ be a Cauchy sequence for the positive solution of $x^2 - 2 = 0$ (i.e., for $\sqrt{2}$). Then α irrational, so also $-\alpha$ irrational (by the second part of 8-1). However $0 = \alpha + (-\alpha)$ is rational.