Solutions for HW 4

Exercise 1.2.14: Solution: For $n \geq 1$ there exist an open set $U_n \supset E$ such that $m(U_n) \leq m^*(E)+\frac{1}{n}$. Let $F = \cap_{n=1}^{\infty} U_n$. Then F is measurable, $E \subset F$ and $m^*(E) \leq m(F) \leq m(U_n) \leq m^*(E) + \frac{1}{n}$ for all $n \geq 1$ implies that $m^*(E) = m(F)$.
Exervise 1.2.15: Solution We first show

$$m(E) = \sup_{F \subset E, F \text{closed}} m(F).$$

Let $\epsilon > 0$. Then there exists an open set $U \supset E^c$ with $m(U \setminus E^c) < \epsilon$. Let $F = U^c$. Then $F \subset E$, F closed and $m(E \setminus F) < \epsilon$. Now $m(E) \leq m(F) + m(E \setminus F) \leq m(F) + \epsilon$ proves the above claim. To get the corresponding formula with compact sets, observe that if F is closed that $K_n = F \cap \{x : |x| \leq n\}$ is compact and $m(F) = \sup_n m(K_n)$. The desired formula follows now easily.
Exercise 1.2.19: Solution: (i)$\Rightarrow$ (ii) For $n \geq 1$ there exist an open set $U_n \supset E$ such that $m(U_n \setminus E) \leq \frac{1}{n}$. Let $G = \cap_{n=1}^{\infty} U_n$. Then G is a G_δ set and $m(G \setminus E) = 0$. The implication (ii)$\Rightarrow$ (i) is obvious as G_δ sets and null sets are measurable. The implication (i)$\Rightarrow$ (iii) follows by finding $G \supset E^c$ as in (ii) such that $m^*(G \setminus E^c) = 0$. Now take $F = G^c$. As the complement of a G_δ set is a F_σ set we are done. The implication (iii)$\Rightarrow$ (i) is obvious as F_σ sets and null sets are measurable.