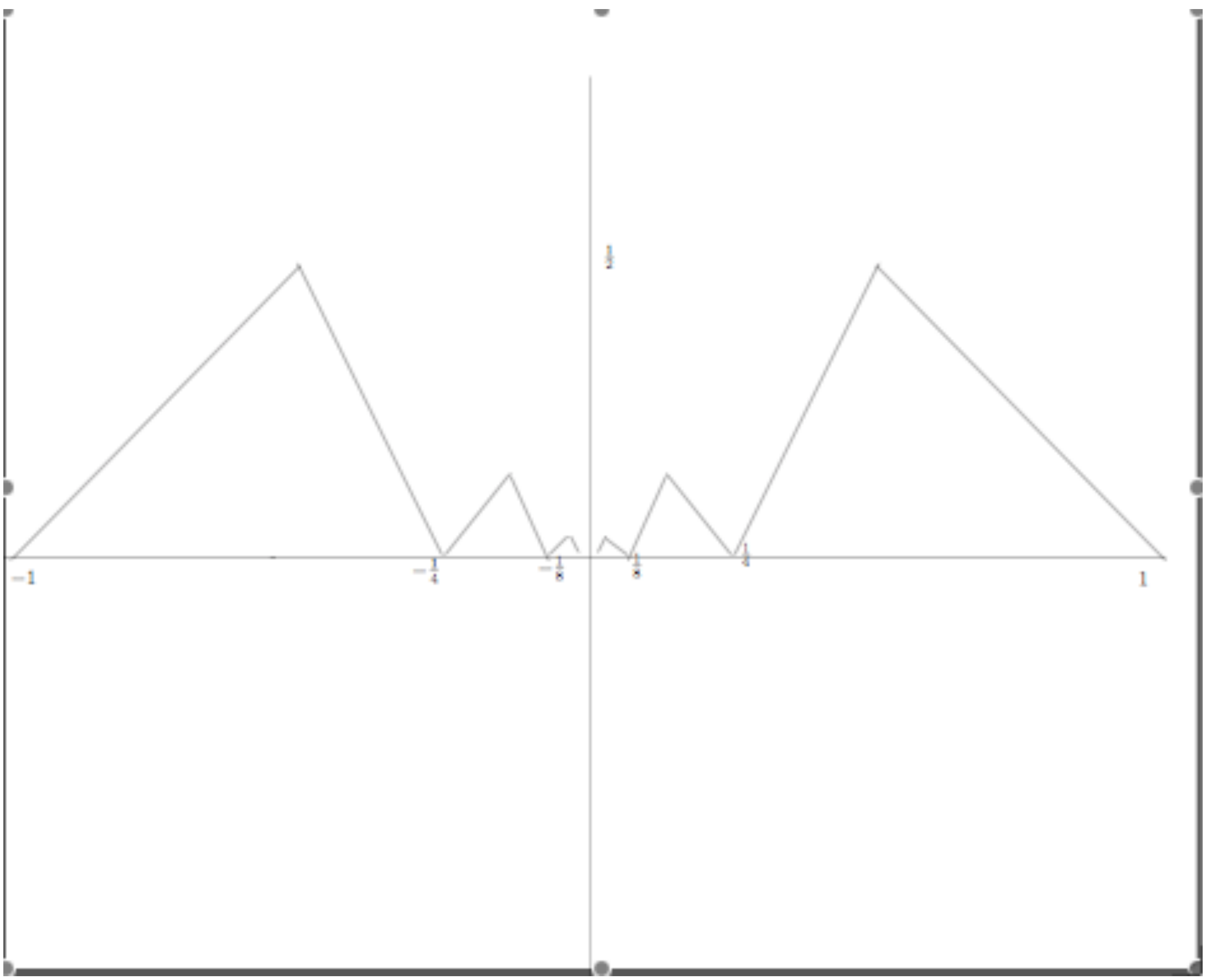
Solutions homework 3.

Problem 1. Solution: Assume $|f(x)| \leq M$ and $|g(x)| \leq N$. Let $\epsilon > 0$. Then there exists $\delta_1 > 0$ such that $|f(x) - f(y)| < \frac{\epsilon}{2N}$ whenever $|x - y| < \delta_1$. Also there exists $\delta_2 > 0$ such that $|g(x) - g(y)| < \frac{\epsilon}{2M}$ whenever $|x - y| < \delta_2$. Let $\delta = \min\{\delta_1, \delta_2\}$. Then $|x - y| < \delta$ implies that

$$\begin{aligned}|f(x)g(x) - f(y)g(y)| &\leq |f(x)g(x) - f(y)g(x)| + |f(y)g(x) - f(y)g(y)| \\ &\leq N|f(x) - f(y)| + M|g(x) - g(y)| < \epsilon.\end{aligned}$$

Problem 2 Solution: Let $M = \max\{|f(x)| : 0 \leq x \leq c\}$. Then by periodicity $|f(x)| \leq M$ for all $x \in \mathbb{R}$. Let $\epsilon > 0$. Then $[0, 2c]$ compact im plies that f is uniformly continuous on $[0, 2c]$. Let $\epsilon > 0$. then there exists $0 < \delta < c$ such that $|x - y| < \delta$ and $x, y \in [0, 2c]$ implies that $f(x) - f(y)| < \epsilon$. Let now $x, y \in \mathbb{R}$ with $|x - y| < \delta$. Assume $x < y$. Then there exists $n \in \mathbb{Z}$ such that $x + nc \in [0, c]$. Then $y < x + \delta < x + c$ implies that $y + nc \in [0, 2c]$. Now $|(x + nc) - (y + nc)| < \delta$ implies that $|f(x) - f(y)| = |f(x + nc) - f(y + nc)| < \epsilon$, so f is uniformly continuous on $\mathbb{R}$.

Problem 3 = Problem 28-1. (a)



(b) If we take $x_n = \frac{1}{4^n}$, the $f(x_n) = 0$, so $\lim_{n\to\infty} \frac{f(x_n) - f(0)}{x_n - 0} = 0$. On the other hand, if we take $y_n = \frac{1}{2\cdot 4^{n-1}}$, then $f(y_n) = \frac{1}{24^{n-1}}$. Hence in this case $\lim_{n\to\infty} \frac{f(y_n)-f(0)}{y_n-0} = 1$, thus f can't be differentiable at 0.

Problem 4 = Problem 28-3.

(1) a is not differentiable at 0, as $\lim_{x\to 0+} \frac{|x|-0}{x-0} = 1$, while $\lim_{x\to 0-} \frac{|x|-0}{x-0} = -1$.
(2) b is not differentiable at 0, as b is not continuous there.
(3) c is differentiable at 0, provided $\beta_0 \neq 0$. In that case the quotient rule applies.

Problem 5. Let $x_n = a + \frac{1}{n}$. Then $\frac{f(x_n)-f(a)}{x_n-a} = n(f(a+\frac{1}{n}) - f(a))$, and the result follows. To get a counterexample it suffices to take $f(x) = |x - \frac{1}{2}|$.