Solutions homework 3.

(1) **Problem 3-7.**

a. We claim

$$\chi_A = \sum_{k=1}^{n} \chi_{A_k}$$

if and only if the sets A_k are mutually disjoint. Assume first that the sets A_k are mutually disjoint. Then $x \in A$ if and only if there exists exactly one k such that $x \in A_k$. This implies that both sides of the above equation are equal to 1 in case $x \in A$. Clearly both sides equal 0 in case $x \notin A$, so the equation holds in this case. Now assume there exists $k \neq l$ such that $A_k \cap A_l \neq \emptyset$. The for $x \in A_k \cap A_l$ we have $\chi_A(x) = 1$, while $\sum_{i=1}^{n} \chi_{A_i}(x) \geq 2$, which is a contradiction.

b. We claim

$$\chi_A = \sum_{k=1}^{n} \chi_{A_k} - \chi_{\cap_{k=1}^{n} A_k}$$

if and only if either $n = 2$ or $n > 2$ and the sets A_k are mutually disjoint. To see this, let $x \in \cap_{k=1}^{n} A_k$. Then $x \in A_k$ for each k, so the right hand side of the equation equals $n-1$ in that case. Hence $1 = n-1$, or $n = 2$ must hold in this case. if $\cap_{k=1}^{n} A_k = \emptyset$, then $n \neq 1$ and we are back in case (a), so that the sets have to mutually disjoint. If $n = 1$, then the equation does not hold. This proves that if the equation holds then either $n = 2$ or $n > 2$ and the sets A_k are mutually disjoint. Conversely if $n = 2$ or $n > 2$ and the sets A_k are mutually disjoint, then it is straightforward to check that the equation holds.

c. The equation in (c) holds if and only if every $x \in A$ belongs to exactly two consecutive A_k's, i.e.., if for every $x \in A$ there exists $1 \leq k \leq n-1$ such that $x \in A_k \cap A_{k+1}$ and $A_k \cap A_{k+1} \cap A_l \cap A_{l+1} = \emptyset$ for all $k \neq l$. For $n = 2$ the condition is that $A_1 \cap A_2 = A_1 \cup A_2$ which implies $A_1 = A_2$. For $n > 2$ we need besides the disjointness condition that $\cup_{k=1}^{n-1}(A_k \cap A_{k+1}) = \cup_{k=1}^{n} A_k$.

(2) **Problem 3-8.**

a. The set equals the union of three parallel line segments of length one in the plane.

b. The set equals a circular cylinder of height one.

c. The set equals the region below the graph of $f(x) = x$ between $x = -1$ and $x = 1$.

(3) **Problem 4-4.** By replacing A by $A \setminus B$ we can assume that B is disjoint with A. If A became finite after replacing it with $A \setminus B$, then the result follows as the union of a finite and a countable set is countable. If A is infinite, we can find a countable subset $C \subset A$. Now $C \cap B$ is countable, so there exists a one-to-one map g from Conto $B \cap C$ (since $B \sim \mathbb{N} \sim B \cup C$). Define now $f : A \to A \cap B$ as follows. If $x \in A \setminus C$ then define $f(x) = x$ and if $x \in C$ then define $f(x) = g(x)$. Then $f : A \to A \cup B$ is one-to-one and onto.

(4) **Problem 4-14.** Let $B \subset A$ be a countable subset. If we can write $B = \cup_{n=1}^{\infty} B_n$ with each B_n infinite, then $A = A \setminus B \cup (\cup_{n=1}^{\infty} B_n)$ is a countable union of infinite sets (if $A \setminus B$ is finite, replace B_1 by $B_1 \cup A \setminus B$). Now $B \sim \mathbb{N}$, so it suffices to write $\mathbb{N}$ as a countable disjoint union of infinite sets. Now $\mathbb{N} \sim \mathbb{N} \times \mathbb{N}$ and $\mathbb{N} \times \mathbb{N} = \cup_{k=1}^{\infty} \{k\} \times \mathbb{N}$ is a countable disjoint union of infinite sets, so we are done. Using the map $f(m, n) = 2^{n-1}(2m-1)$ we can define B_n explicitly, if we write $B = \{b_n | \, n = 1, 2 \cdots\}$. Namely, define $B_n = \{b_{f(m,n)} | \, m = 1, 2, \cdots\}$.