Solutions for HW 3

Exercise 1.1.6: Solution:

(1) Let $\epsilon > 0$. Let $A_1 \subset E \subset B_1$ and $A_2 \subset E \subset B_2$ with $m(B_i \setminus A_i) < \frac{\epsilon}{2}$ for $i = 1, 2$. Then $A_1 \cup A_2 \subset E \cup F \subset B_1 \cup B_2$ and $(B_1 \cup B_2) \setminus (A_1 \cup A_2) \subset (B_1 \setminus A_1) \cup (B_2 \setminus A_2)$ implies that $m((B_1 \cup B_2) \setminus (A_1 \cup A_2)) < \epsilon$, which implies that $E \cup F$ is Jordan measurable. For the intersection note that $A_1 \cap A_2 \subset E \cap F \subset B_1 \cap B_2$ and $(B_1 \cap B_2) \setminus (A_1 \cap A_2) \subset (B_1 \setminus A_1) \cup (B_2 \setminus A_2)$ implies that $m((B_1 \cap B_2) \setminus (A_1 \cap A_2)) < \epsilon$, which implies that $E \cap F$ is Jordan measurable. The proof that $E \setminus F$ is Jordan measurable is similar, using that $A_1 \setminus B_2 \subset E \setminus F \subset B_1 \setminus A_2$. THe symmetrice difference follows now from $E \triangle F = (E \setminus F) \cup (F \setminus E)$.

(2) This is obvious.

(3) Let $\epsilon > 0$. Then there exists elementary sets A_1, A_2, B_1, and B_2 such that $A_1 \subset E \subset B_1$, $A_2 \subset F \subset B_2$ and $m(B_i) - m(A_i) < \frac{\epsilon}{2}$ for $i = 1, 2$. Now $m(A_1 \cup A_2) = m(A_1) + m(A_2)$ implies that $m(E \cup F) \geq m(A_1) + m(A_2) \geq m(B_1) + m(B_2) - \epsilon \geq m(E) + m(F) - \epsilon$. Hence $m(E \cup F) \geq m(E) + m(F)$. From the definition of the outer Jordan measure it follows easily that $m(E \cup F) \leq m(E) + m(F)$, so equality holds.

(4) $E \cup (F \setminus E) = F$ is a disjoint union of elementary sets, so by (3) we have $m(E) + m(F \setminus E) = m(F)$. By (2) we have that $m(F \setminus E) \geq 0$ and the result follows.

(5) See (3).

(6) Follows immediately from $m(B + x) = m(B)$ for a box B.

Exercise 1.2.4: Solution: If $d(E, F) = 0$, then there exist $x_n \in E$ and $y_n \in F$ so that $d(x_n, y_n) \to 0$. Now by compactness we can find a convergent subsequence x_{n_k} which converges to $x_0 \in E$. From $d(x_{n_k}, y_{n_k}) \to 0$ it follows that also $y_{n_k} \to x_0$. Hence $x_0 \in E \cap F$, which is a contradiction. Note this proof works for any metric space. An example where the assertion fails for two closed sets is $E = \mathbb{N}$ and $F = \{n + \frac{1}{n} : n \in \mathbb{N}\}$.

Exercise 1.2.5: Solution: We always have $m_{*,J}(E) \leq m(E)$ by the remark on the bottom of page 22. If $E = \cup_{n=1}^{\infty} B_n$ is union of almost disjoint boxes, then $\cup_{n=1}^{N} B_n \subset E$ implies that $\sum_{n=1}^{N} m(B_n) \leq m_{*,J}(E)$. As this holds for all N we get $\sum_{n=1}^{\infty} m(B_n) \leq m_{*,J}(E)$. This completes the proof as we have $m(E) = \sum_{n=1}^{\infty} m(B_n)$.

Exercise 1.2.6: Solution: Let $E = [0, 1] \setminus \mathbb{Q}$. Then the only open set U contained in E is the empty set, so that $\sup\{m^*(U) : U \subset E, U \text{ open}\} = 0$. If $m^*(E) = 0$, then by subadditivity $m([0, 1]) = 0$, which is a contradiction. Hence $m^*(E) > 0$. Using the measurability of E we can actually see that $m^*(E) = 1$.

Extra Problem: Solution: Assume first E is Jordan measurable. Then for all $\epsilon > 0$ there exist elementary sets $A \subset E \subset B$ with $m(B \setminus A) < \epsilon$. We can assume $A, B \subset [a, b]$. Writing A and B as a disjoint union of intervals we can find a partition $\mathcal{P}$ of $[a, b]$ so that each interval in A and B is finite union of the intervals determined by the partition. Then $A \subset E$ implies that the lower sum of χ_E w.r.t. the partition $\mathcal{P}$ is bigger or equal to $m(A)$ and $E \subset B$ implies that the upper sum of χ_E w.r.t. the partition $\mathcal{P}$ is less or equal to $m(B)$. Hence the difference between the upper and lower sum is less than ϵ and χ_E is Riemann integrable. From $m(A) \leq \int_a^b \chi_E(x)\, dx \leq m(B)$ it also follows that $m(E) = \int_a^b \chi_E(x)\, dx$. Conversely assume now χ_E is Riemann integrable. Let $\epsilon > 0$. Then there exists a partition $\mathcal{P}$ of $[a, b]$ so that the upper sum minus the lower sum is less then ϵ. Let A be the union of the intervals determined by $\mathcal{P}$ which are contained in E. Then $m(A)$ equals the lower sum of χ_E for $\mathcal{P}$. Similarly let B be the union of all the intervals determined by $\mathcal{P}$ which have non-empty intersection with E. Then $E \subset B$ and the upper sum of χ_E for $\mathcal{P}$ equals $m(B)$. Hence $m(B \setminus A) < \epsilon$ and the result follows.