Solutions homework 2.

Page 32 Problem 6: Clearly $d(x,y) \geq 0$ and $d(x,y) = 0$ if and only if $x = y$. Hence property (i) holds. It is also clear that $d(x,y) = d(y,x)$, so it remains to show that the triangle inequality holds. Let $x, y, z \in X$. If $x = y$, then $d(x,y) = 0 \leq d(x,z) + d(z,y)$, since both terms on the right hand side are ≥ 0. If $x \neq y$, then at least one of $x \neq z$ or $z \neq y$ is true. This implies that either $d(x,z) \geq 1$ and/or $d(z,y) \geq 1$. Hence $d(x,z) + d(z,y) \geq 1 = d(x,y)$.

Page 32 Problem 7. From the triangle inequality we get

$$d(x,y) \leq d(x,x') + d(x',y).$$

Another application of the triangle inequality gives

$$d(x',y) \leq d(x',y') + d(y',y).$$

Combining these two inequalities we get

$$d(x,y) \leq d(x,x') + d(x',y') + d(y',y),$$

which gives

$$d(x,y) - d(x',y') \leq d(x,x') + d(y',y).$$

Now interchanging x and x', and y and y' in this last inequality gives (using the symmetry property of the metric)

$$d(x',y') - d(x,y) \leq d(x,x') + d(y',y).$$

Now the last two inequalities combined yield the desired inequality.

Page 43 Problem 12.

a. The triangle inequality gives

$$d(x,y) \leq d(x,x_n) + d(x_n,y).$$

As $d(x,x_n) \to 0$ and $d(x_n,y) \to 0$ it follows that $d(x,y) = 0$ and thus $x = y$.

b. From Problem **7** above we get

$$|d(x,y) - d(x_n,y_n)| \leq d(x,x_n) + d(y,y_n),$$

which implies immediately that $|d(x,y) - d(x_n,y_n)| \to 0$.

c. If $x_n \to x$, then ultimately $d(x,x_n) < 1$, so ultimately $d(x,x_n) = 0$. This implies that the sequence is ultimately constant equal to x. The converse is obviously true in any metric space.

Page 50 Problem 3.

a. Assume (x_n) converges to x. Let $\epsilon > 0$. Then there exists N such that $d(x_n,x) < \frac{\epsilon}{2}$ for all $n \geq N$. Let now $n, m \geq N$. Then $d(x_n, dx_m) \leq d(x_n,x) + d(x,x_m) < \epsilon$. Hence (x_n) is a Cauchy sequence.

b. Assume $(\frac{1}{n})$ converges to $a \in X$. Then $(\frac{1}{n})$ converges to $a \in \mathbb{R}$. By uniqueness of limits it then follows that $a = 0$, which contradicts $a \in X$. Hence $(\frac{1}{n})$ does not converge in X.

c. Take $\epsilon = 1$ in the definition of Cauchy sequence. Then there exists N such that $d(x_n,x_m) < 1$ for all $n, m \geq N$. Hence $d(x_n,x_m) = 0$ for all $n, m \geq N$, i.e., the sequence is ultimately constant and therefore converges.

Page 65 Problem 8(ii). We need to prove that $B_r(c) = \{x \in X : d(x,c) \leq r\}$ is closed. Let $x_n \in B_r(c)$ and assume $x_n \to x$. Then from problem **7** above we get

$$|d(c,x_n) - d(c,x)| \leq d(x,x_n).$$

Hence $d(x_n,c) \to d(x,c)$. Now $d(x_n,c) \leq r$ for all n, implies that $d(x,c) \leq r$.