Solutions homework 2.

Problem 24-4. To prove A_α closed it suffices to show that if $x_n \in A_\alpha$ for all $n \geq 1$ and $x_n \to x$, then $x \in A_\alpha$. Hence let $x_n \in A_\alpha$ and assume assume $x_n \to x$. Then by continuity $f(x_n) \to f(x)$, but $f(x_n) = \alpha$ for all n, so also $f(x) = \alpha$, i.e. $x \in A_\alpha$.

Problem 24-15. Let $A \subset [0,1]$ be dense and assume $f(a) = g(a)$ for all $a \in A$. Then for all $x \in [0,1]$ there exist $a_n \in A$ such that $a_n \to x$. Then by continuity $f(a_n) \to f(x)$ and $g(a_n) \to g(x)$. Now $f(a_n) = g(a_n)$ for all n implies that $f(x) = g(x)$.

Problem 27-3. First Proof: Let $\epsilon > 0$. Then there exists $\delta_1 > 0$ and $\delta_2 > 0$ such that for all $x, y \in A$ with $|x - y| < \delta_1$ we have

$$|f(x) - f(y)| < \frac{\epsilon}{2}$$

and for all $x, y \in A$ with $|x - y| < \delta_2$ we have

$$|g(x) - g(y)| < \frac{\epsilon}{2}.$$

Let $\delta = \min\{\delta_1, \delta_2\}$. Then for all $x, y \in A$ with $|x - y| < \delta$ we have

$$|f(x) + g(x) - f(y) - g(y)| \leq |f(x) - f(y)| + |g(x) - g(y)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Hence $f + g$ is uniformly continuous.
Second Proof. Let $x_n, y_n \in A$ with $x_n - y_n \to 0$. Then by uniform continuity $f(x_n) - f(y_n) \to 0$ and $g(x_n) - g(y_n) \to 0$. Hence also $(f(x_n) + g(x_n)) - (f(y_n) + g(y_n)) \to 0$ and thus by the Theorem from class $f + g$ is uniformly continuous.

Problem 27-4. It was shown in class that $g(x) = x$ is uniformly continuous on $\mathbb{R}$. For $f(x) = \sin x$, notice that $f'(x) = \cos x$ satisfies $|f'(x)| \leq 1$. Hence $|\sin x - \sin y| \leq |x - y|$, which implies that $f(x) = \sin x$ is uniformly continuous on $\mathbb{R}$. To see that the product fg is not uniformly continuous, let $x_n = \pi n$ and $y_n = \pi n + \frac{1}{n}$. Then $f(x_n) = 0$ and $f(y_n) = \cos \pi n \sin \frac{1}{n}$. Hence $|f(x_n) - f(y_n)| = (\pi n + \frac{1}{n}) \sin \frac{1}{n} \to \pi$ as $n \to \infty$, so fg is not uniformly continuous. Here we used that

$$\lim_{n\to\infty} n \sin \frac{1}{n} = \lim_{x\to 0} \frac{\sin x}{x} = 1.$$

Problem 27-5. We will show that if f is uniformly continuous on $(0,1)$, then f is bounded. Let f be uniformly continuous on $(0,1)$ and let $\epsilon = 1$. Then there exists $\delta > 0$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < 1$. Let N be the smallest natural number so that $N\delta > 1$ and define $x_n = n\delta$ for $n = 1, 2, \cdots, N-1$. Then $(0,1) \subset \cup_{n=1}^{N-1}(x_n - \delta, x_n + \delta)$. Hence for all $x \in (0,1)$ there exists n such that $|x_n - x| < \delta$. Thus $|f(x) - f(x_n)| < 1$, so $|f(x)| \leq |f(x_n)| + 1$. Let $M = \max\{|f(x_1)| + 1, \cdots, |f(x_{N-1})| + 1\}$. Then $|f(x)| \leq M$ for all $x \in (0,1)$.