Solutions homework 2.

Problem2-5: Counterexample: Take $T = \{1, 2, 3\}$ and $A_1 = \{0, 2\}, A_2 = \{1, 2\}$ and $A_3 = \{0, 1\}$.

Problem 2-6. Let first $x \in A_1$. Assume $x \notin \cap_{n\in\mathbb{N}} A_n$. Then there exists at least one positive integer such that $x \notin A_n$. Since $x \in A_1$, there is a smallest natural number n such that $x \in A_n$ but $x \notin A_{n+1}$, i.e., there exists a natural number n such that $x \in A_n \setminus A_{n+1}$. This shows

$$A_1 \subset \left[\bigcup_{n\in\mathbb{N}} A_n \setminus A_{n+1}\right] \cup \left[\bigcap_{n\in\mathbb{N}} A_n\right]. \tag{1}$$

Now assume $x \in [\cup_{n\in\mathbb{N}} A_n \setminus A_{n+1}] \cup [\cap_{n\in\mathbb{N}} A_n]$. Then $x \in \cup_{n\in\mathbb{N}} A_n \setminus A_{n+1}$ or $x \in \cap_{n\in\mathbb{N}} A_n$. In the first case $x \in A_n \setminus A_{n+1}$ for soem n, so $x \in A_n \subset A_1$ implies then that $x \in A_1$. In the second case $x \in A_n$ for all $n \geq 1$, so certainly $x \in A_1$. Hence we equality in (1). To prove the sets are pairwise disjoint, let $n < m$. Then $x \in A_n \setminus A_{n+1}$ implies $x \notin A_{n+1}$. As $m \geq n + 1$ we have $A_m \subset A_{n+1}$ and thus $x \notin A_m$. This implies that $[A_n \setminus A_{n+1}] \cap [A_m \setminus A_{m+1}] = \emptyset$. If $x \in A_n \setminus A_{n+1}$ then $x \notin A_{n+1}$, which shows $x \notin \cap_n A_n$. Hence $[A_n \setminus A_{n+1}] \cap [\cap_n A_n] = \emptyset$.

Problem 2-12.

(a) $n \sim n$, since $n - n = 0$ is even. If $n \sim m$, then there exists an integer k such that $n - m = 2k$. Then $m - n = 2(-k)$ is also even, so $m \sim n$. Now assume that $n \sim m$ and $m \sim p$. Then there exist integers k, l such that $n - m = 2k$ and $m - p = 2l$. Hence $n - p = n - m + m - p = 2k + 2l = 2(k + l)$ is even, so $n \sim p$. Thus $\sim$ is an equivalence relation.

(b) Not an equivalence relation, e.g. it is not reflexive $2 \nsim 2$.

(c) Note $(a, b) \sim (c, d)$, if $a - b = c - d$. Now it is easy to check that $\sim$ is an equivalence relation.

(d) If $A \neq \emptyset$, then $A \nsim A$, so it is not reflexive.

Problem 3-3. Yes, $f(A \cup B) = f(A) \cup f(B)$. Proof: Let first $y \in f(A \cup B)$. Then there exists $x \in A \cup B$ such that $y = f(x)$. Now $x \in A$ or $x \in B$, so $y = f(x) \in f(A)$ or $y = f(x) \in f(B)$, i.e., $y \in f(A) \cup f(B)$. Conversely, if $y \in f(A) \cup f(B)$, then $y \in f(A)$ or $y \in f(B)$. If $y \in f(A)$, then there exists $x \in A$ such that $y = f(x)$ and if $y \in f(B)$, then there exists $x \in B$ such that $y = f(x)$. Either way, there exists $x \in A \cup B$ such that $y = f(x)$. Thus $y \in f(A \cup B)$.

Problem 3-6. For $g \circ f$ to have an inverse it needs to be one-to-one and onto. We claim that $g \circ f$ is one-to-one and onto if and only if f is one-to-one and the restriction $g_{|f(A)}$ of g to the range of f is one-to-one and onto. Note f does not need to be onto, and there is no restriction on the values of g on $B \setminus f(A)$ in case f is not onto. To prove the claim, note first that the range $g \circ f(A)$ of $g \circ f$ equals $g(f(A))$, from which the necessity and sufficiency part of the onto condition is obvious. For the one-to-one part of the claim assume first that f is one-to-one and the restriction $g_{|f(A)}$ of g to the range of f is one-to-one. Let $x \neq y$ in A. Then $f(x) \neq f(y)$ in $f(A)$ (as f is one-to-one) and thus $g(f(x)) \neq g(f(y))$ (as $g_{|f(A)}$ is one-to-one), i.e., $g \circ f(x) \neq g \circ f(y)$ and thus $g \circ f$ is one-to-one. Conversely, if f is not one-to-one, then there exists $x \neq y$ in A with $f(x) = f(y)$. Then $g \circ f(x) = g(f(x)) = g(f(y)) = g \circ f(y)$ and $g \circ f$ is not one-to-one. Similarly if $g_{|f(A)}$ is not one-to-one, then $g \circ f$ is not one-to-one.