Solutions for HW 2

Exercise 1.1.3: Solution: Let $c = m'([0,1)^d)$. Then $[0,1)^d$ is a disjoint union of n^d translates of $[0,\frac{1}{n})^d$, and thus $m'([0,\frac{1}{n})^d) = \frac{1}{n^d}c$. This implies now that $m'([0,\frac{k_1}{n_1}) \times \cdots [0,\frac{k_d}{n_d})) = (\prod_{i=1}^d \frac{k_i}{n_i})c$ for any positive integers k_i and n_i by taking $n = n_1 \times \cdots \times n_d$ and writing the box as a finite disjoint union of cubes. Now m' is monotone, so by approximating positive real numbers from below and above by rationals we find that $m'(B) = m(B)c$ for any box B, which proves the claim.

Exercise 1.1.4: Solution: Write $E_1 = \cup_{i=1}^{n_1} B_i$ and $E_2 = \cup_{j=1}^{n_2} B'_j$ as disjoint union of boxes. Then $E_1 \times E_2 = \cup_{i,j} B_i \times B'_j$ is a disjoint union of boxes in $\mathbb{R}^{d_1+d_2}$ with $|B_i \times B'_j| = |B_i| \times |B'_j|$, from which the result follows:

$$m^{d_1+d_2}(E_1 \times E_2) = \sum_{i,j} |B_i| \times |B'_j| = \sum_i |B_i| \times \sum_j |B'_j| = m^{d_1}(E_1) m^{d_2}(E_2).$$

Exercise 1.1.5: Solution: (1) $\Rightarrow$ (2) Let $\epsilon > 0$. Then (1) implies that there exist elementary sets A, B with $A \subset E \subset B$ such that $m(B) - m(A) < \epsilon$. Now $B = A \cup (B \setminus A)$ is a disjoint union of elementary sets, so $m(B \setminus A) = m(B) - m(A) < \epsilon$.
(2) $\Rightarrow$ (3) Let A, B as in (2). Then $A \triangle E = E \setminus A \subset B \setminus A$ implies that (3) holds.
(2) $\Rightarrow$ (1) follows immediately from $m(B) - m(A) = m(B \setminus A)$.
Remains to show (3) $\Rightarrow$ (2). Let $\epsilon > 0$. Find an elementary A such that $m^{*,J}(A \triangle E) < \epsilon$. Then by definition of the outer Jordan measure there exist an elementary set C with $A \triangle E \subset C$ with $m(C) < \epsilon$. Let $B = A \cup C$ and $A' = A \setminus C$. Then $A' = A \setminus C \subset A \setminus (A \triangle E) \subset A \setminus (A \setminus E) = A \cap E \subset E$. Now $B \setminus A' = C$ implies that (2) holds.