Solutions homework 12.

(1) **Problem 18-2** Let $A \subset B$ and assume $x \in A'$. Then there exists a sequence $\{a_n\}$ in A such that $a_n \to x$ and $a_n \neq x$. Hence there exists the sequence $\{a_n\}$ in B such that $a_n \to x$ and $a_n \neq x$. This implies that $x \in B'$. Note one can also use the definition as follows: For all $\epsilon > 0$ the intersection $N(x, \epsilon) \cap A$ contains an element $a \neq x$. Hence for all $\epsilon > 0$ the intersection $N(x, \epsilon) \cap B$ contains the element $a \neq x$. For $A = \mathbb{R}$ and $B = \mathbb{Q}$ we have $A' \subset B'$ (they are actually equal), but A is not a subset of B.

(2) **Problem 18-3** As $A, B \subset A \cup B$ we get from problem 18-2 that $A', B' \subset (A \cup B)'$ and thus $A' \cup B' \subset (A \cup B)'$. For the reverse inclusion, let $x \in (A \cup B)'$. Then there exists a sequence $\{x_n\}$ in $A \cup B$ such that $x_n \to x$ and $x_n \neq x$. Since we have only two sets, we have a subsequence $\{x_{k_n}\}$ of $\{x_n\}$ such that $x_{k_n} \in A$ for all k_n or $x_{k_n} \in B$ for k_n. As $x_{k_n} \to x$ we get that either $x \in A'$ or $x \in B'$. hence $x \in A' \cup B'$.

(3) **Problem 18-4** As $A_n \subset \cup_{n \in \mathbb{N}} A_n$ for all n, we get $A'_n \subset (\cup_{n \in \mathbb{N}} A_n)'$ for all n. Hence

$$\cup_{n \in \mathbb{N}} A'_n \subset (\cup_{n \in \mathbb{N}} A_n)' .$$

For a counterexample of the reverse inclusion, write $\mathbb{Q} = \{r_n : n = 1, 2 \cdots\}$. Put $A_n = \{r_n\}$. Then $A'_n = \emptyset$, so $\cup_{n \in \mathbb{N}} A'_n = \emptyset$. On the other hand $(\cup_{n \in \mathbb{N}} A_n)' = \mathbb{Q}' = \mathbb{R}$.