Solutions homework 11.

(1) **Problem 17-3** Let $A \subset \mathbb{R}$ consist of isolated points only. Then for all $a \in A$ there exists an open interval $I_a = (a - r_a, a + r_a)$ such that $A \cap I_a = \{a\}$. In particular, if $a, b \in A$ and $a \neq b$, then $r_a \leq |a-b|$ and $r_b \leq |a-b|$. Let $J_a = (a - \frac{r_a}{2}, a + \frac{r_a}{2})$ for all $a \in A$. We claim that $\{J_a : a \in A\}$ is disjoint collection of open intervals with $a \in J_a$ for all $a \in A$. Let $x \in J_a \cap J_b$ and assume that $a \neq b$. Then $|a-b| \leq |a-x| + |x-b| < \frac{r_a}{2} + \frac{r_b}{2} \leq \frac{|a-b|}{2} + \frac{|a-b|}{2} = |a-b|$, i.e., $|a-b| < |a-b|$, which is a contradiction. Now for each $a \in A$ we can choose $q_a \in J_a \cap \mathbb{Q}$. Then $a \neq b$ implies that $q_a \neq q_b$ as $J_a \cap J_b = \emptyset$. Hence $A \sim \{J_a : a \in A\} \sim \{q_a : a \in A\}$, i.e., A is equivalent to a subset of $\mathbb{Q}$. This implies directly that A is either finite or countable, as every subset of a countable set is either finite or countable.

(2) **Additional Problem 1**

a. Let A be nowhere dense and $B \subset A$. Then for any open interval I we can find an open interval $J \subset I$ with $A \cap J = \emptyset$. Then $B \subset A$ implies that also $B \cap J = \emptyset$. Hence J is nowhere dense.

b. Let I be an open interval and $c \in \mathbb{R}$.. Then $I + (-c)$ is an open inerval, so there exist an open interval $J \subset I + (-c)$ with $J \cap A = \emptyset$. Now $J + c$ is an open interval with $J + c \subset I$ and $J + c \cap A + c = \emptyset$. Hence $A + c$ is nowhere dense.

c. if $c = 0$, then $cA = \{0\}$, which is nowhere dense. Assume therefore $c \neq 0$. Let I be an open interval. Then $\frac{1}{c}I$ is an open interval, so there exist an open interval $J \subset \frac{1}{c}I$ with $J \cap A = \emptyset$. Now cJ is an open interval with $cJ \subset I$ and $cJ \cap cA = \emptyset$. Hence cA is nowhere dense.

d. For all $a \in A$ let I_a be an open interval such that $I_a \cap A = \{a\}$. Let I be an open interval. If $I \cap A = \emptyset$, then we can take $J = I$. Assume therefore there exist $a \in A$ with $a \in I$. If we write $I = (\alpha, \beta)$. Then we can take $J = (\alpha, a) \subset I$ with $J \cap A = \emptyset$.

(3) **Additional Problem 2** Let A be a set of the first category and assume $B \subset A$. Then we can write $A = \cup_{n=1}^{\infty} A_n$, where each A_n is nowhere dense. Define $B_n = B \cap A_n$. Then by Additional Problem 1 a. above, we have that each B_n is nowhere dense. Moreover $\cup_{n=1}^{\infty} B_n = \cup_{n=1}^{\infty} (A_n \cap B) = (\cup_{n=1}^{\infty} A_n) \cap B = A \cap B = B$. Hence B is of the first category.

(4) **Additional Problem 3**

a. Note $A = \cap_{n=1}^{\infty} \left(r_k - \frac{\epsilon}{2^k}, r_k + \frac{\epsilon}{2^k}\right)^c$. Hence it suffices to show that for each k the set $\left(r_k - \frac{\epsilon}{2^k}, r_k + \frac{\epsilon}{2^k}\right)^c$ is closed, as the intersection of closed sets is closed. Now $\left(r_k - \frac{\epsilon}{2^k}, r_k + \frac{\epsilon}{2^k}\right)^c = \left(-\infty, r_k - \frac{\epsilon}{2^k}\right] \cap \left[r_k + \frac{\epsilon}{2^k}, \infty\right)$ is closed, since it is finite union of closed sets. Hence A is closed.

b. To prove A nowhere dense, let I be an open interval. Then there exists $k \geq 1$ such that $r_k \in I$. Take $J = I \cap \left(r_k - \frac{\epsilon}{2^k}, r_k + \frac{\epsilon}{2^k}\right)$. Then J is an open interval with $J \subset I$ and $J \cap A = \emptyset$.