Solutions for HW 11

Problem 1: Solution:

a. Let $f \in L^\infty(X,\mu)$. Then $\int_X |f|^r \, d\mu \leq \mu(X)\|f\|_\infty^r < \infty$, so $f \in L^r(X,\mu)$. Now let $1 \leq p < r$ and $f \in L^r(X,\mu)$. Put $s = \frac{r}{p}$. Then $s > 1$. Let $1 < t < \infty$ such that $\frac{1}{s} + \frac{1}{t} = 1$. Now apply Hölder's inequality to get

$$\int |f|^p \, d\mu \leq \left(\int |f|^{ps} \, d\mu \right)^{\frac{1}{s}} \mu(X)^{\frac{1}{t}} = \|f\|_r^{\frac{1}{s}} \mu(X)^{\frac{1}{t}} < \infty.$$

To see that the inclusions are strict, note that $f(x) = x^{-\frac{1}{r}}$ is not in $L^r([0.1])$, but is in $L^p([0,1])$ for all $1 \leq p < r$. Also $f(x) = x^{-\frac{1}{s}} \in L^r([0,1])$ for $s > r$, but not in $L^\infty([0.1])$.

b. Let $f \in L^\infty \cap L^1$. Then $\int |f|^p \, d\mu = \int |f||f|^{p-1} \, d\mu \leq \|f\|_\infty^{p-1} \int |f| \, d\mu < \infty$. Now let $f \in L^p$. Let $E = \{x : |f(x)| \geq 1\}$. Then $\mu(E) \leq \int |f|^p \, d\mu < \infty$. Let $h = f\chi_E$. Then by part **a.** $h \in L^1(E,\mu)$, so $h \in L^1(X,\mu)$. Now $g = f - h$ satisfies $|g(x)| < 1$ for all x.

Problem 2: Solution: From Hölder's inequality with $p = q = 2$ we have

$$\left(\int_{[0,1]} x f(x) \, dx \right)^2 \leq \left(\int_0^1 x^2 \, dx \right) \left(\int_{[0,1]} |f(x)|^2 \, dx \right).$$

Problem]3: Solution: Let $\epsilon > 0$.Then by Egorov's Theorem there exists E_ϵ with $m(E_\epsilon^c) < \epsilon^q$ such that f_n converges uniformly to 0 on E_ϵ. Let N be such that $|f_n(x)| < \frac{\epsilon}{m(E)}$ for all $n \geq N$. Then

$$\int |f_n| \, dx = \int_{E_\epsilon^c} |f| + \int_{E_\epsilon} |f| \leq \|f_n \chi_{E_\epsilon^c}\|_p m(E_\epsilon^c)^{\frac{1}{q}} + \frac{\epsilon}{m(E)} m(E_\epsilon) < 2\epsilon$$

for all $n \geq N$.

Problem 4: Solution:

(1) Observe first that also $|g| \leq M$ a.e. and thus $|(g_n - g)f|^p \leq 2^p M^p |f|^p$ a.e. It follows now from the Dominated Convergence Theorem that

$$\int |(g_n - g)f|^p \, dx \to 0.$$

(2) Using the triangle inequality we have $\|g_n f_n - fg\|_p \leq \|f_n g_n - f g_n\|_p + \|(g_n - g)f\|_p \leq M\|f - f_n\|_p + \|(g_n - g)f\|_p \to 0$.