Solutions homework 10.

(1) **Problem 14-1** Assume $\sum a_k$ converges. Then by the Comparison Test 13.5 we get that $\sum b_k$ converges, which is a contradiction. Hence $\sum a_k$ diverges.

(2) **Problem 14-2** First proof: By the Cauchy condensation test $\sum 2^k a_{2^k}$ converges. Let $\epsilon > 0$. Then there exists k_0 such that $2^k a_{2^k} < \frac{\epsilon}{2}$ for all $k \geq k_0$ by 13.2. Now let $n \geq 2^{k_0}$. Then there exists $k_1 \geq k_0$ such that $2^{k_1} \leq n < 2^{k_1+1}$. Then $na_n \leq 2^{k_1+1}a_{2^{k_1}} < 2 \cdot 2^{k_1} a_{2^{k_1}} < 2 \cdot \frac{\epsilon}{2} = \epsilon$. The converse is false. Take e.g. $a_k = \frac{1}{k \ln k}$. Then $\lim k a_k = \lim \frac{1}{\ln k} = 0$, but $\sum a_k$ diverges (as shown in class, or see text).
Second proof. Let $\epsilon > 0$. Then by the Cauchy Convergence Tests there exists k_0 such that $\sum_{k=m}^n a_k < \frac{\epsilon}{2}$ for all $n > m \geq k_0$. Now let $n \geq k_0$. Then $\frac{n+k_0}{n} \leq 2$. Now $n \cdot a_{n+k_0} \leq \sum_{k_0}^{n+k_0} a_k < \frac{\epsilon}{2}$ implies that $(n+k_0)a_{n+k_0} \leq \frac{n+k_0}{n} \cdot \frac{\epsilon}{2} = \epsilon$, i.e. $n \cdot a_n < \epsilon$ for all $n \geq 2k_0$.

(3) **Problem 14-4** (**a**) $0 < \frac{1}{k(k+1)} \leq \frac{1}{k^2}$ implies by the Comparison Test that the series converges, since $\sum \frac{1}{k^2}$ converges.
(**d**) For $k \geq 3$ we have $\ln k \geq \ln 3 = a > 1$. Hence $\frac{1}{(\ln k)^k} < \frac{1}{a^k}$ for $k \geq 3$. Hence by the comparison test with the geometric series $\sum (\frac{1}{a})^k$ the series converges.
(**e**) Observe that $\frac{1}{k^{1+\frac{1}{k}}} = \frac{1}{k} \cdot \frac{1}{k^{\frac{1}{k}}}$. Now $k^{\frac{1}{k}} \to 1$ as $k \to \infty$ (Theorem 11.9), so there exists k_0 such that $k^{\frac{1}{k}} < 2$ for all $k \geq k_0$. This implies that $\frac{1}{k^{1+\frac{1}{k}}} > \frac{1}{2} \cdot \frac{1}{k}$ for all $k \geq k_0$. Hence by Problem 14.1 and the fact that the harmonic series diverges it follows that $\frac{1}{k^{1+\frac{1}{k}}}$ diverges.
(**f**) Use the Cauchy condensation test to see that the original series converges if and only if the series $\sum \frac{1}{k(\ln k + \ln 2)}$ converges. Now $k(\ln k + \ln 2) \leq 2k \ln k$ implies that this series is greater or equal than $\frac{1}{2} \sum \frac{1}{k \ln k}$ which diverges (done in class). Therefore the original series diverges.

(4) **Problem 14-6** The inequality $0 < \frac{a_k}{1+a_k} \leq a_k$ shows that $\sum \frac{a_k}{1+a_k}$ converges whenever $\sum a_k$ converges. The converse is also true. If $\sum \frac{a_k}{1+a_k}$ converges, then $\frac{a_k}{1+a_k} \to 0$ as $k \to \infty$. This implies that also $a_k \to 0$ as $k \to \infty$ (look at the graph of $f(x) = \frac{x}{1+x}$). This implies that $1 + a_k < 2$ for all $k \geq k_0$. Hence $a_k \leq 2 \cdot \frac{a_k}{1+a_k}$ for all $k \geq k_0$. This shows that $\sum a_k$ converges whenever $\sum \frac{a_k}{1+a_k}$ converges.