Solutions for HW 11

Exercise 1.6.49: Solution:

(iv) One checks easily that $|\sqrt{x}-\sqrt{y}|/|x-y| = 1/(\sqrt{x}+\sqrt{y})$ on $(0,1]$, which implies that $f(x)=\sqrt{x}$ is not Lipschitz on $[0,1]$.Moreover from calculus we have for $x \neq 0$ that $f'(x) = \frac{1}{2\sqrt{x}}$. From the Fundamental theorem of Calculus we have that

$$f(x) = f(\epsilon) + \int_\epsilon^1 f'(t)\,dt.$$

letting $\epsilon \downarrow 0$ we get by the Monotone Convergence theorem

$$f(x) = \int_0^1 f'(t)\,dt,$$

which implies that f is absolutely continuous.

(v) Already seen that the Cantor function f is continuous on $[0,1]$. As $[0,1]$ is compact this implies that f is uniformly continuous. if f would be absolutely continuous, then $f'(x)=0$ a.e. implies that f is constant, which is a contradiction. Hence f is not absolutely continuous.

Problem 1: Solution: Since absolutely continuous functions are of bounded variation we get $\int_a^b |F'(x)|\,dx \leq \|F\|_{TV[a,b]}$. For the other direction, let $a = x_0, \cdots x_n = b$ be a partition of $[a,b]$. Then we get

$$\begin{aligned}\sum_{i=1}^n |F(x_i)-F(x_{i-1})| &= \sum_{i=1}^n |\int_{x_{i-1}}^{x_i} F'(x)\,dx| \\ &\leq \sum_{i=1}^n \int_{x_{i-1}}^{x_i} |F'(x)|\,dx = \int_a^b |F'(x)|\,dx.\end{aligned}$$

This implies that $\|F\|_{TV[a,b]} \leq \int_a^b |F'(x)|\,dx$ and the proof is complete.

Problem 2: Solution: As $a>0$ we have that $|F(x)| \leq x^a \to 0$ as $x \to 0$, so F is continuous at 0. Also for $x \neq 0$ we have $F'(x) = ax^{a-1}\sin\frac{1}{x^b} - bx^{a-b-1}\cos\frac{1}{x^b}$. Hence $|F'(x)| \leq ax^{a-1} + bx^{a-b-1} \in L^1[0,1]$, so $F' \in L^1[0,1]$. For $\epsilon > 0$ the function F' is continuous on $[\epsilon,1]$, and thus bounded. This implies that F is Lipschitz on $[\epsilon,1]$ and thus absolutely continuous on $[\epsilon,1]$. From observation in class it follows that F is absolutely continuous

Problem 3: Solution: Since f_n is absolutely continuous and $f_n(0)=0$ we have $f_n(x) = \int_0^x f_n'(t)\,dt$. Hence

$$|f_n(x) - f_m(x)| \leq \int_0^1 |f_n'(t) - f_m'(t)|\,dt \to 0$$

uniformly in x as $n,m \to \infty$. Thus there exists $f:[0,1]\to\mathbb{R}$ such that f_n converges uniformly to f. To prove f is absolutely continuous, observe that $\{f_n'\}$ is a Cauchy sequence in $L^1([0,1])$. Hence there exists $g \in L^1([0,1])$ such that $\|f_n' - g\|_1 \to 0$. This implies that

$$\int_0^x f_n'(t)\,dt \to \int_0^x g(t)\,dt$$

as $n \to \infty$ for every $x \in [0,1]$ (even uniformly in x). Hence $f(x) = \int_0^x g(t)\,dt$, which implies that f is absolutely continuous.