Solutions homework 1.

(1) Prove that $[-1;1)$ is not compact by using the definition of a compact set (to get credit for the problem, use the definition and not any theorems about compact sets).

Proof: Let $O_n = (-1, 1 - \frac{1}{n})$. Then $[-1;1) \subset \cup_n O_n$, but $[-1;1)$ can't be covered by $\cup_{n=1}^N O_n$ for any N, as $1 - \frac{1}{2N} \notin \cup_{n=1}^N O_n$.

(2) What is an interior point? Prove that $\frac{1}{4}$ is an interior point of $(0;2]$.

Proof: p is an interior point of E if there exists $\epsilon > 0$ such that $(p-\epsilon, p+\epsilon) \subset E$. Let $\epsilon = \frac{1}{4}$. Then $(\frac{1}{4} - \epsilon, \frac{1}{4} + \epsilon) = (0, \frac{1}{2}) \subset (0;2]$.

(3) Let $a_1 = \sqrt{6}$ and $a_{n+1} = \sqrt{6 + a_n}$ for $n \geq 1$.

a. Show that $a_n \leq 3$ for all $n \geq 1$.
For $n = 1$ we have $a_1 = \sqrt{6} \leq \sqrt{9} = 3$. Assume now that $a_n \leq 3$. Then $a_{n+1} = \sqrt{6 + a_n} \leq \sqrt{6+3} = 3$. Hence by induction $a_n \leq 3$ for all $n \geq 1$.

b. Show that $\{a_n\}$ is an increasing sequence.
For $n = 1$ we get $a_2 = \sqrt{6 + \sqrt{6}} \geq \sqrt{6} = a_1$. Assume now that $a_{n+1} \geq a_n$. Then $a_{n+2} = \sqrt{6 + a_{n+1}} \geq \sqrt{6 + a_n} = a_{n+1}$. It follows by induction that $\{a_n\}$ is an increasing sequence.

c. Explain why $\{a_n\}$ converges.
Every bounded increasing sequence converges (to the supremum of the sequence).

d. Determine the value of $\lim_{n\to\infty} a_n$.
Let $a = \lim_{n\to\infty} a_n$. Then $a_{n+1} = \sqrt{6 + a_n}$ implies that $a = \sqrt{6 + a}$. Hence $a^2 = 6 + a$, so $a = 3$ or $a = -2$. This implies $a = 3$, as $a \geq a_1 \geq \sqrt{6}$.

(4) Complete the table below indicating Int(E) (the interior of E), the set of isolated points of E, and whether E is open, closed, both, or neither. An answer of "open" or "closed" in the next to last column will mean that you think E is "open and not closed" or "closed and not open" respectively. You do not need to show work on this problem.

E	Int(E)	Isol. pts. of E	Open? or Closed?	Compact
$(-1, 1]$	$(-1, 1)$	none	neither	no
$(0, \infty) \cap \mathbb{Q}$	$\emptyset$	none	neither	no
$\mathbb{R}$	$\mathbb{R}$	none	both	no
$[2, \infty)$	$(2, \infty)$	none	closed	no

(5) Suppose p is in the closure of two sets A and B.

a. Must p be in the closure of $A \cup B$? Justify your answer.
Yes. Every neighborhood of p will contain $q \in A$ with $q \neq p$ and observe $q \in A$ implies $q \in A \cup B$.

b. Must p be in the closure of $A \cap B$? Justify your answer.
No. Take e.g. $A = (0, 1)$, $B = (1, 2)$, and $p = 1$.