Solutions for HW 9

Problem 1.4: 8

a. First solution: Let $f : \mathbb{N} \to A$ and $g : \mathbb{N} \to B$ be one-to-one and onto. Then $h = g \circ f^{-1} : A \to B$ is one-to-one and onto.
Second solution: Let $A = \{a_1, a_2, \cdots\}$ and $B = \{b_1, b_2, \cdots\}$. Then define $h : A \to B$ by $h(a_k) = b_k$. Then h is one-to-one and onto.

b. Let $f : \mathbb{N} \to A$ and $g : \mathbb{A} \to B$ be one-to-one and onto. Then $h = g \circ f : \mathbb{N} \to B$ is one-to-one and onto. Hence B is countably infinite.

Problem 1.4: 20. For each $I \in \mathcal{F}$ we can find a rational number $q_I \in I$ by Theorem 1.18. If $I \neq J$, then $I \cap J = \emptyset$ implies that $q_I \neq q_J$. Hence the map $I \to q_I$ is a one-to-one map from $\mathcal{F}$ into $\mathbb{Q}$. As $\mathbb{Q}$ is countable, it follows that $\mathcal{F}$ is countable.

Problem 1.4: 21. Let S be an infinite set and pick $x_1 \in S$. Then $S \setminus \{x_1\} \neq \emptyset$ implies we can pick $x_2 \in S \setminus \{x_1\}$. Assuming that we have picked $x_1, \cdots, x_{n-1}$ in S, we have that $S \setminus \{x_1, \cdots, x_{n-1}\} \neq \emptyset$, so we can pick an $x_n \in S \setminus \{x_1, \cdots, x_{n-1}\}$. This way we obtain a countably infinite subset $\{x_n : n = 1, 2, \cdots\}$ in S.