Solutions for HW 28

Problem 4.2: 10 Define the polynomial P by $P(x) = x^7 + x^5 + x^3 + 1$. Then $P(x) > 0$ for $x > ge0$. Moreover P has at least one real zero. Assume that P has two real zero's $a < b < 0$. Then from $P(a) = P(b) = 0$ it follows from Rolle's Theorem that $P'(c) = 0$ for some c with $a < c < b$. But $P'(c) = c^2(7c^4 + 5c^2 + 3) > 0$, since $c < 0$. Contradiction.
Problem 4.2: 18 Let $x < y$ in $\mathbb{R}$. Then there exists $x < c < y$ such that $f(y) - f(x) = f'(c)(y - x)$. Hence if $|f'(c)| \leq M$ for all c, then $|f(y) - f(x)| \leq M|y - x|$. Hence if $\epsilon > 0$ is given. then take $\delta = \frac{\epsilon}{M}$. Then $|y - x| < \delta$ implies $|f(y) - f(x)| < \epsilon$.
Problem 4.2: 26 To compute $f'(0)$ we use the definition:

$$\lim_{h \to 0} \frac{f(h)}{h} = \lim_{h \to 0} \frac{1}{2} + h \sin\left(\frac{1}{h}\right) = \frac{1}{2}.$$

On the other hand for $x \neq 0$ we have $f'(x) = \frac{1}{2} + 2x \sin(\frac{1}{x}) - \cos(\frac{1}{x})$. Hence for $x_n = \frac{1}{(2n+1)\pi}$ we have $f'(x_n) = -\frac{1}{2}$. Hence $f'(x) < 0$ on an interval around x_n for all n. Hence f is decreasing on an interval around x_n for all n.
Problem 4.2: 27 Note first that $f'(x) = 1 + \cos x \geq 0$ and $f'(x) = 0$ only for $x_n = (2n+1)\pi$ for $n \in \mathbb{Z}$. This implies that f is strictly increasing on each interval (x_{n-1}, x_n), which implies that f is strictly increasing on each interval $[x_{n-1}, x_n]$ and thus on $\mathbb{R}$.