Solutions for HW 24

Problem 3.4: 2a,b,c,e

a. Let $f(x) = 23x + 47$ and let $\epsilon > 0$. Then take $\delta = \frac{\epsilon}{23}$. Then for $|x - y| < \delta$ we have $|f(x) - f(y)| = 23|x - y| < 23\delta = \epsilon$.

b. Observe first that for $x, y \in [0, 3]$ we have $|f(x) - f(y)| \leq |x^2 - y^2| + 2|x - y| \leq (|x| + |y|)(|x - y|) + 2|x - y| \leq 8|x - y|$. Hence for $\epsilon > 0$ we take $\delta = \frac{\epsilon}{8}$. Then $|x - y| < \delta$ and $x, y \in [0, 3]$ implies that $|f(x) - f(y)| < \epsilon$.

c. Observe first that

$$\left| \frac{4}{x^2} - \frac{4}{y^2} \right| \leq 4 \cdot \frac{|x - y|\,(|x| + |y|)}{x^2 y^2} \leq 20\,|x - y|\,.$$

Therefore for $\epsilon > 0$ we choose $\delta = \frac{\epsilon}{20}$.

e. Let $x_n = n$ and $y_n = n + \frac{1}{n}$. Then $x_n - y_n \to 0$, but $|f(y_n) - f(x_n)| = 3n + \frac{3}{n} + \frac{1}{n^3} \to \infty$ as $n \to \infty$. Hence f is not uniformly continuous.

Problem 3.4: 5 Let $f : I \to \mathbb{R}$ be uniformly continuous and $\{x_n\}$ a Cauchy sequence in I. Then for $\epsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - f(y)| < \epsilon$ for all $x, y \in I$ with $|x - y| < \delta$. For this $\delta > 0$ we can find N such that $|x_n - y_n| < \delta$ for all $n \geq N$. Hence $|f(x_n) - f(y_n)| < \epsilon$ for all $n \geq N$, which shows that $\{f(x_n)\}$ is a Cauchy sequence.

Problem 3.4: 6

a. Let $\epsilon > 0$. Then there exists $\delta_1 > 0$ such that $|f(x) - f(y)| < \frac{\epsilon}{2}$ for all $x, y \in I$ with $|x - y| < \delta_1$. Also there exists $\delta_2 > 0$ such that $|g(x) - g(y)| < \frac{\epsilon}{2}$ for all $x, y \in I$ with $|x - y| < \delta_2$. Let $\delta = \min\{\delta_1, \delta_2\}$. Then $|f(x) + g(x) - f(y) - g(y)| < \epsilon$ for all $x, y \in I$ such that $|x - y| < \delta$.

b. Take $I = [0, \infty)$ and f, g defined by $f(x) = g(x) = x$.

c. If both f and g are uniformly bounded on I, then fg will be again uniformly continuous whenever f and g are uniformly continuous. **Proof:** Let M be such that $|f(x)| \leq M$ and $|g(x)| \leq M$ for all $x \in I$. Let $\epsilon > 0$. Then by taking the minmum of the two delta's we can find a $\delta > 0$ such that $|f(x) - f(y)| < \frac{\epsilon}{2M}$ and $|g(x) - g(y)| < \frac{\epsilon}{2M}$ if $|x - y| < \delta$. Then

$$\begin{aligned} |f(x)g(x) - f(y)g(y)| &= |f(x)g(x) - f(y)g(x) + f(y)g(x) - f(y)g(y)| \\ &\leq M\,|f(x) - f(y)| + M\,|g(x) - g(y)| < \epsilon \end{aligned}$$

if $|x - y| < \delta$.

d. A sufficient condition is that there exists $m > 0$ such that $|f(x)| \geq m$ on I. The proof of this is similar to the the proof that $\frac{1}{f}$ is continuous at c in case f is continuous at c and $f(c) \neq 0$. Let $\epsilon > 0$. Then there exists $\delta > 0$ such that $|f(x) - f(y)| < \epsilon m^2$ for all $x, y \in I$ with $|x - y| < \delta$. Now $|\frac{1}{f(x)} - \frac{1}{f(y)}| = \frac{|f(x) - f(y)|}{|f(x)f(y)} \leq \frac{|f(x) - f(y)|}{m^2} < \epsilon$ for all $x, y \in I$ with $|x - y| < \delta$.