Problem 3.3: 19 If J is an open interval and $f : J \to \mathbb{R}$ is strictly monotone, then $f(J)$ is an open interval. The reason is that, if e.g. f is strictly increasing and if $f(c) = d$ for some $c \in J$, then we can always find $a < c < d$ in J. Thus $f(a) < f(c) < f(b)$ and f can't have a max or min at c. As this is true for any point $c \in J$, we see that the interval $f(J)$ can't contain its endpoints.

Problem 3.3: 25 Consider first f defined by $f(x) = x^{\frac{1}{q}}$. Then for q even f has domain $[0, \infty)$ and is the inverse on the strictly increasing continuous function $g : [0, \infty) \to [0, \infty)$ defined by $g(x) = x^q$. That g is continuous follows from the fact that it is a polynomial and if $0 \le x < y$, then $0 \le \frac{x}{y} < 1$ implies $0 \le (\frac{x}{y})^n < 1$ or $0 \le x^n < y^n$. If q is odd, then we can define a strictly increasing function $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = x^n$. To see that g is strictly increasing let $x < y$. If $x \ge 0$, then the prove is as before. If $x < 0$ and $y > 0$, then $x^n < 0$ and $y^n > 0$, so $x^n < y^n$. Hence it remains to check the case that $x < y \le 0$, but then $0 \le -y < -x$. Then by the previous part $0 \le (-y)^n < (-x)^n$, now $(-y)^n = -y^n$ and $(-x)^n = -x^n$ as n is odd. This implies that $0 \le -y^n < -x^n$ or $x^n < y^n \le 0$. In this case f defined by $f(x) = x^{\frac{1}{q}}$ is function from $\mathbb{R}$ to $\mathbb{R}$. This shows that the function f defined by $f(x) = x^{\frac{1}{q}}$ is continuous for every x where it is defined and any positive integer q. Now consider f defined by $f(x) = x^{\frac{p}{q}}$, where p is non-zero integer and q is a positive integer, which have greatest common denominator equal to 1. As above the domain is $[0, \infty)$ in case q is even and $\mathbb{R}$ incase q is odd. Define h by $h(x) = x^{\frac{1}{q}}$ with the domains as above and g by $g(x) = x^p$. The domain of g is $\mathbb{R}$ if $p \ge 1$ and $\mathbb{R} \setminus \{0\}$ in case $p \le -1$. In both cases h and g are continuous on their domains and therefore $f = g \circ h$ is continuous for every x for which it is defined.

Problem 3.3: 27 Observe first that $f(x) = 3(x^2 - 2x) - 1 = 3(x-1)^2 - 4$. Hence $(-\infty, 1]$ is the largest interval on which f is strictly decreasing. To find f^{-1} we solve $y = 3(x-1)^2 - 4$ for x to get $x = 1 - \sqrt{(\frac{y+4}{3})}$ (note we take the negative root to get a solution which is ≤ 1). Hence $f^{-1} : [-4, \infty) \to (-\infty, 1]$ is given by $f^{-1}(x) = 1 - \sqrt{(\frac{x+4}{3})}$.

Problem 3.3: 28 By Problem 25 we know that the function defined by $x \mapsto x^{\frac{1}{5}}$ is continuous for all x. Also $x \mapsto \sin x$ and $x \mapsto 3x$ are continuous everywhere, so is thus its composition $x \mapsto \sin(3x)$. This implies that $x \mapsto x^3 + x62\sin(3x)$ is continuous everywhere, so by the composition theorem the numerator is continuous everywhere. The denominator is a polynomial and thus continuous everywhere. Moreover the denominator is never 0. Hence the function h is continuous everywhere by the theorem about quotients of continuous functions. Summarizing we needed: polynomials are continuous everywhere, the sine function is continuous everywhere, composition of continuous functions are continuous, sums, products and quotients of continuous functions are continuous (as long as the denominator of the quotient is $\neq 0$).