Solutions for HW 21

Problem 3.3: 6 Define $f : [\frac{\pi}{6}, \frac{\pi}{2}] \to \mathbb{R}$ by $f(x) = x^2 - \sin x$. Then f is continuous and $f(\frac{\pi}{6}) = \frac{\pi^2}{36} - \frac{1}{2} = -.22\cdots < 0$ and $f(\frac{\pi}{2}) = \frac{\pi^2}{4} - 1 = 1.4\cdots > 0$. Hence by the Intermediate Value Theorem f has a zero on $[\frac{\pi}{6}, \frac{\pi}{2}]$.

Problem 3.3: 7

a. Let $P(x) = a_n x^n + \cdots + a_0$ be a polynomial of odd degree n. Multiplying P by -1, if necessary, we can assume that $a_n > 0$. Then $\lim_{x\to\infty} P(x) = x^n(a_n + \frac{a_{n-1}}{x} + \cdots + \frac{a_0}{x^n}) = \infty$, so there exists $b > 0$ with $P(b) > 0$. Similarly $\lim_{x\to-\infty} P(x) = -\infty$, so there exists $a < 0$ with $P(a) < 0$. Now apply the Intermediate Value Theorem to P restricted to $[a, b]$ to get $c \in [a, b]$ with $P(c) = 0$.

b. Assume that P is as in a). Let $w \in \mathbb{R}$. Then $\lim_{x\to\infty} P(x) = \infty$ implies that there exists $b > 0$ with $P(b) > w$. Similarly $\lim_{x\to-\infty} P(x) = -\infty$, so there exists $a < 0$ with $P(a) < w$. Now apply the Intermediate Value Theorem to P restricted to $[a, b]$ to get $c \in [a, b]$ with $P(c) = w$. Note another way to prove is to put $Q(x) = P(x) - w$. Then Q is a polynomial of odd degree, so by part a0 there exists c with $Q(c) = 0$, i.e., $P(c) = w$.

Problem 3.3: 8 Define $f : [2\pi, \frac{5\pi}{2}) \to \mathbb{R}$ by $f(x) = \tan x - x$. Then f is continuous, $f(2\pi) = -2\pi < 0$ and $\lim_{x\to\frac{5\pi}{2}-} = \infty$, so there exists b with $2\pi < b < \frac{5\pi}{2}$ such that $f(b) > 0$. By the Intermediate Value Theorem f has a zero at some $x > 2\pi$.

Problem 3.3: 9 Note $f(a) \geq a$ and $f(b) \leq b$. Define g by $g(x) = f(x) - x$. then either $g(a) = 0$, $g(b) = 0$, or $g(a) > 0$ and $g(b) < 0$ in which case $g(x) = 0$ for some x with $a < x < b$.