Solutions for HW 17

Problem 2.3:2 $x_{2n} = \frac{2n}{4n+3} = \frac{2}{4+\frac{3}{n}} \to \frac{2}{4} = \frac{1}{2}$ and $x_{2n+1} = \frac{-(2n+1)}{4n+5} = \frac{-2-\frac{1}{n}}{4+\frac{5}{n}} \to -\frac{1}{2}$.

Problem 2.3:3

a. Let $\{x_n\} = \{1, 2, 3, 1, 2, 3, \cdots\}$, or $x_{3n} = 3$, $x_{3n-1} = 2$, and $x_{3n-2} = 1$ for all $n \geq 1$.
b. Let $x_n = (-1)^n n$.
c. Let $\{x_n\} = \{1, -1, 3, 2, -2, 3, \cdots\}$, or $x_{3n} = 3$, $x_{3n-1} = -n$, and $x_{3n-2} = n$ for all $n \geq 1$.
d. Same sequence as part d).
e. Same sequence as part b).

Problem 2.3:6 Let $L = \lim_{n\to\infty} a_{2n} = \lim_{n\to\infty} a_{2n-1}$. Then for $\epsilon > 0$ there exists N_1 such that $|a_{2n} - L| < \epsilon$ for all $n \geq N_1$ and there exists N_2 such that $|a_{2n-1} - L| < \epsilon$ for all $n \geq N_2$. Let $N = \max\{2N_1, 2N_2 - 1\}$. Then for $n \geq N$ we have $|a_n - L| < \epsilon$ (two cases either $n = 2k$ with $k \geq N_1$ or $n = 2k - 1$ with $k \geq N_2$).

Problem 2.3:8

a. Applying the formula twice we find

$$x_{2n} = \frac{1}{1 + x_{2n-1}} = \frac{1}{1 + \frac{1}{1+x_{2n-2}}} = \frac{1 + x_{2n-2}}{2 + x_{2n-2}}.$$

Hence $a_n = \frac{1+a_{n-1}}{2+a_{n-1}}$. Note $a_1 = x_2 = \frac{1}{2}$. We shall prove directly by induction that $\{a_n\}$ is increasing. For $n = 1$ we have that $a_1 = \frac{1}{2} \leq \frac{3}{5} = a_2$. Assume we know that $a_{n-1} \leq a_n$. Then $a_{n+1} = \frac{1+a_n}{2+a_n} = 1 - \frac{1}{2+a_n}$. Now $a_{n-1} \leq a_n$ implies

$$a_n = 1 - \frac{1}{2 + a_{n-1}} \leq 1 - \frac{1}{2 + a_n} = a_{n+1}.$$

Hence $\{a_n\}$ is increasing and bounded above by 1. Note $a_n + a_n^2 \leq 1$ follows now immediately from $a_n \leq 1 - \frac{1}{2+a_n}$ (multiply both sides by $2 + a_n$).

b. Note $b_n = \frac{1}{1+a_{n-1}}$, which implles immediately that $\{b_n\}$ is decresasing and bounded below by 0.
c. From part a) we know that $\lim a_n = L$ exists and satisfies $L = \frac{1+L}{2+L}$. Solving for L we find $L^2 + L - 1 = 0$ and thus $L = \frac{-1+\sqrt{5}}{2}$

(the other root is < 0). From $b_n = \frac{1}{1+a_{n-1}}$ we see that $\lim b_n = \frac{1}{1+L}$, but $L^2+L-1 = 0$ implies that $L = \frac{1}{1+L}$, so $\lim b_n = \lim a_n$. From problem 6 it follows now that $\lim x_n$ exists and equals L.