Solutions for HW 16

Problem 2.4: 1 Easy to see that $x_2 = 8, x_3 = 20, x_4 = 40$ and $x_5 = 70$. A formula for x_n is given by $x_n = \frac{1}{3}(n+2)(n+1)n$. To prove this, one can use induction. For $n = 1$ we have $\frac{1}{3}(1+2)(1+1)1 = 2 = x_1$. Assume now that the formula holds for $n-1$. Then $x_n = \frac{1}{3}(n+1)n(n-1) + n(n+1) = \frac{1}{3}n(n+1)(n-1+3) = \frac{1}{3}(n+2)(n+1)n$. Hence the formula holds for n.

Problem 2.4: 6 Let $\lim a_n = a \neq 0$. Then $a_n \neq 0$ for all $n \geq N$ for some N. Then $\lim b_n = \lim a_n b_n \lim \frac{1}{a_n} = \frac{1}{a} \lim a_n b_n$.

Problem 2.4: 7

a. Take $a_n = \frac{1}{n^2}$ and $b_n = n$.
b. Take $a_n = \frac{1}{n}$ and $b_n = n^2$.
c. Take $a_n = \frac{1}{n}$ and $b_n = \pi n$.

Problem 2.4: 10

a. $\sqrt{n+1} - \sqrt{n} = \frac{(n+1)-n}{\sqrt{n+1}+\sqrt{n}} = \frac{1}{\sqrt{n+1}+n} \to 0$.
b. Let $a_n = \frac{n!}{n^n}$. Then $\frac{a_{n+1}}{a_n} = (n+1)\frac{n^n}{(n+1)^{n+1}} = \frac{1}{(1+\frac{1}{n})^n} \leq \frac{1}{2}$. Hence $a_{n+1} \leq (\frac{1}{2})^{n-1} a_1 \to 0$.
c. Let $a_n = \frac{1 \cdot 3 \cdots (2n-1)}{n!}$. Then $\frac{a_{n+1}}{a_n} = \frac{2n+1}{n+1} \geq \frac{3}{2}$. Hence $a_{n+1} \geq (\frac{3}{2})^{n-1} a_1 \to \infty$.