Solutions for HW 15

Problem 2.2:11 Let $a_1 = a > 0$. Then $a_n = \sqrt{2a_{n-1}}$ for all $n \geq 2$. We claim that the sequence $\{a_n\}$ is increasing when $a \leq 2$ and decreasing when $a \geq 2$. This follows from $a_n = \sqrt{2a_{n-1}} \geq a_{n-1}$ if and only if $2a_{n-1} \geq a_{n-1}$ if and only if $2 \geq a_{n-1} \geq 0$. When $a < 2$ we have that the increasing sequence is bounded above by 2, while when $a > 2$, then the decreasing sequence $\{a_n\}$ is bounded below by 2. Hence in both cases $\lim a_n = L$ exists and satisfies $L = \sqrt{2L}$. Hence $L^2 = 2L$ or $L(L-2) = 0$. As $L = 0$ is not possible, we get that $L = 2$.

Problem 2.2:24 By iterating $|x_{n+1} - x_n| \leq r\,|x_n - x_{n-1}|$ we get that

$$|x_{n+1} - x_n| \leq r\,|x_n - x_{n-1}| \leq r^2\,|x_{n-1} - x_{n-2}| \leq \cdots \leq r^{n-1}\,|x_2 - x_1|\,.$$

Now let $m > n$. Then

$$\begin{aligned} |x_m - x_n| &\leq |x_m - x_{m-1}| + \cdots + |x_{n+1} - x_n| \\ &\leq \left(r^{m-2} + \cdots + r^{n-1}\right) |x_2 - x_1| \\ &\leq \frac{r^n}{1-r} |x_2 - x_1| < \epsilon, \end{aligned}$$

for all $n \geq N$.

Problem 2.2:31

a. First proof: From Theorem 2.14 we know $\lim \sqrt[n]{4} = 1$ and $\lim \sqrt[n]{n} = 1$. Hence $\lim \sqrt[n]{4n} = \lim \sqrt[n]{4} \cdot \lim \sqrt[n]{n} = 1$. Second proof: Let $\sqrt[n]{4n} = 1 + t_n$. Then $t_n \geq 0$ and $4n = (1+t_n)^n \geq \frac{1}{2}n(n-1)t_n^2$ for $n \geq 2$. Hence $0 \leq t_n \leq \sqrt{\frac{8}{n-1}}$ for all $n \geq 2$. Hence by the Squeeze theorem $t_n \to 0$.

b. First proof: Use Theorem 2.14 to get $\lim \sqrt[n]{n^2} = \lim \sqrt[n]{n} \cdot \lim \sqrt[n]{n} = 1$. Second proof: Let $\sqrt[n]{n^2} = 1 + t_n$. Then $t_n \geq 0$ and $n^2 = (1+t_n)^n \geq \frac{1}{6}n(n-1)(n-2)t_n^3$ for $n \geq 3$. Hence $0 \leq t_n \leq \sqrt[3]{\frac{6n}{(n-1)(n-2)}}$ for $n \geq 3$. Hence by the Squeeze theorem $t_n \to 0$.

c. Let $10^{\frac{1}{n^2}} = 1 + t_n$. Then $t_n \geq 0$ and $10 = (1+t_n)^{n^2} \geq n^2 t_n$. Hence $0 \leq t_n \leq \frac{10}{n^2}$. Hence by the Squeeze theorem $t_n \to 0$.