Solutions for HW 14

Problem 2.2:12

Note first that

$$\left|\frac{n}{n+3}-\frac{m}{m+3}\right| \le \left|\frac{n}{n+3}-1\right|+\left|\frac{m}{m+3}-1\right| = \left|\frac{3}{n+3}+\frac{3}{m+3}\right|.$$

Let $\epsilon > 0$. Then there exists N such that $\frac{3}{n+3} < \epsilon$ for all $n \ge N$. Hence $\left|\frac{n}{n+3}-\frac{m}{m+3}\right| < \frac{\epsilon}{2}+\frac{\epsilon}{2} = \epsilon$ for all $n, m \ge N$.

Problem 2.2:14 Let $x_n = (-1)^n$. Then $|x_{n+1} - x_n| = 2$ for n. Hence by taking $\epsilon < 2$ we can never have that $|x_n - x_m| < \epsilon$ for all $n, m \ge N$.

Problem 2.2:18

a. Denote $x_n = \sum_{k=1}^n \frac{\sin k}{k^2}$. Then

$$|x_n - x_m| = \left|\sum_{k=n+1}^m \frac{\sin k}{k^2}\right| \le \sum_{k=n+1}^m \frac{1}{k^2}.$$

From the example right after Definition 2.11 we know that given $\epsilon > 0$ there exists N such that $\sum_{k=n+1}^m \frac{1}{k^2} < \epsilon$ for all $m \ge n \ge N$. Hence $\{x_n\}$ is Cauchy and thus convergent.

b. Replace $\sin k$ by $(-1)^k$ in part a. The rest is identical.

c. Denote $x_n = \sum_{k=1}^n \frac{(-1)^k}{k}$. Then

$$|x_n - x_m| = \left|\sum_{k=n+1}^m \frac{(-1)^k}{k}\right| = \left|\frac{1}{n+1} - \frac{1}{n+2} + \cdots \pm \frac{(-1)^m}{m}\right| < \frac{1}{n+1}.$$

Hence $\{x_n\}$ is Cauchy and thus convergent. Alternate solution: Since $x_{2n} = (-1+\frac{1}{2})+(-\frac{1}{3}+\frac{1}{4})\cdots+(-\frac{1}{2n-1}+\frac{1}{2n})$ and each term in brackets is < 0, we see that $\{x_{2n}\}$ is a decreasing sequence. By writing $x_{2n} = -1 + (\frac{1}{2} - \frac{1}{3}) + \cdots + \frac{1}{2n}$ we also see that x_{2n} is bounded below by -1. Hence $\lim x_{2n}$ exists. Similarly $\lim x_{2n-1}$ exists, since $\{x_{2n-1}\}$ is increasing and bounded above by 0. From $x_{2n} - x_{2n-1} \to 0$ as $n \to \infty$ it follows that $\lim x_{2n} = \lim x_{2n-1}$, and this implies $\lim x_n$ exists.

Problem 2.2:26

a. Observe $r^2 + \cdots + r^{n+1} = r^2(1 + \cdots + r^{n-1}) = \frac{r^2 - r^{n+2}}{1-r} \to \frac{r^2}{1-r}$.

b. Observe $\sum_{k=1}^n r^{2k} = \sum_{k=1}^n (r^2)^k = \frac{r^2 - r^{2n+2}}{1-r^2} \to \frac{r^2}{1-r^2}$.

c. Observe $\sum_{k=0}^n (1-r)^k = \frac{1-(1-r)^{n+1}}{1-(1-r)} \to \frac{1}{r}$.