Solutions for HW 13

Problem 2.2:3

Claim: $\{x_n\}$ is decreasing for $n \geq 2$. To see this, note first that

$$x_{n+1} - x_n = \frac{5}{x_n} - \frac{x_n}{2} = \frac{10 - x_n^2}{2x_n}.$$

Hence $x_{n+1} < x_n$ if and only if $x_n^2 > 10$. We prove this by induction. For $n = 2$ we have $x_2^2 = \frac{49}{4} > 10$. Assume now $x_n^2 > 10$ holds. Then $x_{n+1}^2 - 10 = (\frac{x_n}{2} + \frac{5}{x_n})^2 - 10 = (\frac{x_n}{2} - \frac{5}{x_n})^2 > 0$. Hence $\{x_n\}$ is decreasing for $n \geq 2$ and bounded below by $\sqrt{10}$. Hence $x = \lim x_n$ exists, and satisfies $x \geq \sqrt{10}$ and $x = \frac{x}{2} + \frac{5}{x}$, or $x^2 = 10$. Hence $x = \sqrt{10}$.

Problem 2.2:5

Obviously $\{x_n\}$ is increasing (as $x_{n+1} - x_n > 0$). To show $\{x_n\}$ bounded, observe that $x_n \leq \sum_{k=1}^{n} (\frac{2}{3})^k \leq \frac{1}{(1-\frac{2}{3})} = 3$. Hence $\{x_n\}$ is bounded and increasing, thus convergent.

Problem 2.2:7

First we prove by induction that $0 < x_n \leq 1$ for all $n \geq 1$. For $n = 1$ this is obvious. Assume it holds for n. Then $0 \leq \frac{(x_n^3+2)}{3} \leq \frac{1+2}{3} = 1$ implies that also $0 < x_{n+1} \leq 1$. Hence the sequence is bounded. To see that $\{x_n\}$ is increasing, observe that $x_{n+1} - x_n = \frac{(x_n^3+2)}{3} - x_n = \frac{x_n^3 - 3x_n + 2}{3} = \frac{(x_n-1)^2(x_n+1)}{3} > 0$ (or prove by induction that $x_{n+1} \geq x_n$). Hence the limit $x = \lim x_n$ exists and satisfies $0 \leq x \leq 1$ and $x = (x^3 + 2)/3$. Hence $(x - 1)^2(x + 2) = 0$ and thus $x = 1$.

Problem 2.2:10

We prove by induction that $1 \leq a_n \leq 3$ and that $\{a_n\}$ is increasing. Obviously $1 \leq a_1 \leq 3$ Assume $1 \leq a_n \leq 3$. Then $\frac{1}{3} \leq \frac{1}{a_n} \leq 1$, so $2 \leq a_{n+1} \leq \frac{8}{3}$. In particular $1 \leq a_{n+1} \leq 3$. Next we prove that $\{a_n\}$ is increasing. First observe $a_1 = 1 \leq 2 = a_2$. Assume now that $a_{n-1} \leq a_n$. Then $\frac{1}{a_n} \leq \frac{1}{a_{n-1}}$ implies that $a_{n+1} = 3 - \frac{1}{a_n} \geq 3 - \frac{1}{a_{n-1}} = a_n$. Hence by induction we see that $\{a_n\}$ is increasing. Let $a = \lim a_n$. Then $a = 3 - \frac{1}{a}$, and $1 \leq a \leq 3$. Solving for a we find $a = \frac{3+\sqrt{5}}{2}$.