Solutions for HW 11

Problem 2.1:1 $a_1 = \frac{1\cdot 3}{2!} = \frac{3}{2}$, $a_2 = \frac{1\cdot 3\cdot 5}{4!} = \frac{5}{8}$, $a_3 = \frac{1\cdot 3\cdot 5\cdot 7}{6!} = \frac{7}{48}$, $a_4 = \frac{1\cdot 3\cdot 5\cdot 7\cdot 9}{8!} = \frac{9}{384} = \frac{3}{128}$, and $a_5 = \frac{1\cdot 3\cdot 5\cdot 7\cdot 9\cdot 11}{10!} = \frac{11}{3840}$. Note

$$\frac{a_{n+1}}{a_n} = \frac{1\cdot 3\cdots(2n+3)}{1\cdot 3\cdots(2n+1)}\frac{(2n)!}{(2n+2)!} = \frac{2n+3}{(2n+1)(2n+2)}.$$

Problem 2.1:2 $a_1 = 3$, $a_2 = 3+2 = 5$, $a_3 = 5+3 = 8$, $a_4 = 8+4 = 12$, $a_5 = 12+5 = 17$. To find a formula we iterate the recursion:

$$a_n = a_{n-1}+n = a_{n-2}+(n-1)+n = \cdots = a_1+2+\cdots+n = 1+1+2+\cdots+n = 2+\frac{1}{2}n(n+1).$$

Problem 2.1:3 The range equals $\{-2, -\sqrt{2}, 0, \sqrt{2}, 2\}$

Problem 2.1:5 Suppose $|x_n| \le M$ for all $n \ge 1$ and $|y_n| \le N$ for all $n \ge 1$. Then $|x_n \pm y_n| \le M+N$ for all $n \ge 1$ by the triangle inequality and $|x_n y_n| \le MN$ for all $n \ge 1$.

Problem 2.1:7

a. Take $x_n = n$.
b. Take $x_n = (-1)^n$.
c. Take $x_n = -\frac{1}{n}$.
d. Take $x_n = \frac{(-1)^n}{n}$.
e. Take $x_n = -n$.
f. Take $x_n = n(-1)^n$.
g. Take $x_n = n$ for n even and $x_n = -1$ for n odd..

Problem 2.1: 8

a.
$$\frac{a_{n+1}}{a_n} = \frac{2^{n+1}}{2^n}\frac{n!}{(n+1)!} = \frac{2}{n+1} \le 1,$$
so the sequence is decreasing.

b.
$$\frac{a_{n+1}}{a_n} = \frac{(n+4)}{(6n+5)}\frac{(6n-1)}{(n+3)} = \frac{6n^2+23n-4}{6n^2+23n+15} \le 1,$$
so the sequence is decreasing.

c.
$$\frac{a_{n+1}}{a_n} = \frac{\sqrt[n+1]{4}}{\sqrt[n]{4}} = 4^{\frac{1}{n+1}-\frac{1}{n}} = 4^{-\frac{1}{n(n+1)}} \le 1,$$
so the sequence is decreasing.