Solutions for HW 10

Problem 1.4: 17

a. Define $f : (a, b) \to \mathbb{R}$ by $f(x) = \frac{x - \frac{a+b}{2}}{(x-a)(b-x)}$. then one can check that f is one-to-one onto. As $\mathbb{R}$ is uncountable, it follows that (a, b) is uncountable. Note: one can also mimic the proof that $[0, 1]$ is uncountable, but be sure to use closed sub-intervals!

b. As $S = (a, b) \cap \mathbb{Q}$, it is a subset of a countably infinite set and thus countable. It remains to show S is infinite. To see this, note that there exists a rational number q_1 such that $a < q_1 < b$. Then there exists a rational number q_2 such that $a < q_2 < q_1$. Repeating this we find for each $n \geq 2$ a rational number q_n such that $a < q_n < q_{n-1}$. This provides the infinite subset $\{q_n : n = 1, 2, \cdots\}$ of S. Hence S is infinite.