AND STILL ONE MORE PROOF OF THE RADON–NIKODYM THEOREM.

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Many different proofs of the Radon–Nikodym theorem have appeared in textbooks. Usually either they use the Hahn decomposition theorem for signed measures (see [1], [2],or [3]) or they employ Hilbert space techniques [4], [6]. Best known in this latter category is von Neumann's proof (see [4, Theorem 6.10]), which uses the Riesz representation theorem of bounded linear functionals on a Hilbert space. In [5] a new proof was given, where the Radon–Nikodym derivative was constructed by maximizing certain quadratic functionals. For the special case that $0 \leq \nu \leq \mu$ we feel that our proof is much more transparent. For the general case we revert to von Neumann's approach, except that we can completely avoid the use of Hilbert spaces or the Hahn decomposition theorem.

Lemma 1. *Let ν and μ be finite measures on $(X, \mathcal{B})$ satisfying $0 \leq \nu \leq \mu$ on $\mathcal{B}$. Then there exists a measurable function f_0 with $0 \leq f_0 \leq 1$ such that $\nu(E) = \int_E f_0 \, d\mu$ for all E in $\mathcal{B}$.*

Proof. Let $H = \{f : f \text{ measurable}, 0 \leq f \leq 1, \int_E f \, d\mu \leq \nu(E) \text{ for all } E \in \mathcal{B}\}$. Note that $H \neq \emptyset$, since 0 belongs to H. Moreover, when $f_1, f_2 \in H$, then $\max\{f_1, f_2\} \in H$. Indeed, if $A = \{x : f_1(x) \geq f_2(x)\}$ and $B = A^c$, then

$$\begin{aligned} \int_E \max\{f_1, f_2\} \, d\mu &= \int_{E\cap A} \max\{f_1, f_2\} \, d\mu + \int_{E\cap B} \max\{f_1, f_2\} \, d\mu \\ &= \int_{E\cap A} f_1 \, d\mu + \int_{E\cap B} f_2 \, d\mu \leq \nu(E \cap A) + \nu(E \cap B) = \nu(E). \end{aligned}$$

Let $M = \sup\{\int f \, d\mu : f \in H\}$. Then $0 \leq M < \infty$, so there exist functions f_n in H with $f_1 \leq f_2 \leq \ldots \leq 1$ such that $\int f_n \, d\mu > M - \frac{1}{n}$. Let $f_0 = \lim f_n$. Then f_0 is measurable. By the Monotone Convergence Theorem, $f_0 \in H$ and $\int f_0 \, d\mu \geq M$. Hence $\int f_0 \, d\mu = M$. To complete the proof we show that $\nu(E) = \int_E f_0 \, d\mu$.

Suppose $\nu(E) > \int_E f_0 \, d\mu$ for some E in $\mathcal{B}$. Then we can write $E = E_0 \cup E_1$, where $E_1 = \{x \in E : f_0(x) = 1\}$ and $E_0 = E \setminus E_1$. It now follows from

$$\nu(E) = \nu(E_0) + \nu(E_1) > \int_E f_0 \, d\mu = \int_{E_0} f_0 \, d\mu + \mu(E_1) \geq \int_{E_0} f_0 \, d\mu + \nu(E_1)$$

that also $\nu(E_0) > \int_{E_0} f_0 \, d\mu$. Let $F_n = \{f_0 < 1 - \frac{1}{n}\} \cap E_0$. Then $F_n \uparrow E_0$, so there exists n_0 such that $\nu(F_{n_0}) > \int_{F_{n_0}} f_0 \, d\mu$. Now $f_0 + \epsilon\chi_{F_{n_0}} \in H$ for $\epsilon > 0$ sufficiently small, but $\int f_0 + \epsilon\chi_{F_{n_0}} \, d\mu = M + \epsilon\mu(F_{n_0}) \geq M + \epsilon\nu(F_{n_0}) > M$, which is a contradiction. Hence $\nu(E) = \int_E f_0 \, d\mu$ for all E in $\mathcal{B}$. □

Our proof of the following theorem is patterned on von Neumann's proof of the same theorem, except that we use Lemma 1 instead of the Riesz representation theorem of bounded linear functionals on Hilbert spaces. For completeness we include the proof.

Theorem 2 (**Lebesgue–Radon–Nikodym**). *Let ν and μ be finite measures on $(X,\mathcal{B})$. Then there exist D in $\mathcal{B}$ with $\mu(D)=0$ and a nonnegative μ–integrable function f_0 such that*

$$\nu(E)=\nu(E\cap D)+\int_E f_0\,d\mu$$

for all E in $\mathcal{B}$.

Proof. Let $\lambda=\mu+\nu$. Then $0\le\nu\le\lambda$, so by Lemma 1 there exists g with $0\le g\le 1$ such that $\nu(E)=\int_E g\,d\lambda$ for all E in $\mathcal{B}$. It follows that $\mu(E)=\int_E(1-g)\,d\lambda$ for all E from $\mathcal{B}$. Let $D=\{x:g(x)=1\}$. Then $\mu(D)=\int_D 0\,d\lambda=0$. Now $\nu(E)=\int_E g\,d\nu+\int_E g\,d\mu$ implies that $\int_E(1-g)\,d\nu=\int_E g\,d\mu$ for all E belonging to $\mathcal{B}$. Hence $\int(1-g)\phi\,d\nu=\int g\phi\,d\mu$ for all nonnegative simple functions ϕ, and thus $\int(1-g)f\,d\nu=\int gf\,d\mu$ for all nonnegative measurable functions f. Taking $f=(1+g+\ldots+g^n)\chi_E$ we learn that

$$\int_E(1-g^{n+1})\,d\nu=\int_E g(1+g+\cdots+g^n)\,d\mu$$

for all E in $\mathcal{B}$ and all $n\ge 1$. Since $0\le g(x)<1$ on D^c, it follows from the Monotone Convergence Theorem that

$$\begin{aligned}\nu(E\cap D^c)&=\lim\int_{E\cap D^c}(1-g^{n+1})\,d\nu=\lim\int_{E\cap D^c}g(1+g+\cdots+g^n)\,d\mu\\&=\int_{E\cap D^c}g(1-g)^{-1}\,d\mu=\int_E f_0\,d\mu,\end{aligned}$$

where $f_0=g(1-g)^{-1}\chi_{D^c}$ and the proof is complete. □

Remark. Instead of using von Neumann's ideas one can also follow the ideas of [5] more closely. Observe first that in our lemma we really prove that the essential supremum f_0 of the set H is an element of H. If we introduce f_m as the essential supremum of the set

$$H_m=\{f: f \text{ measurable}, 0\le f\le m, \int_E f\,d\mu\le\nu(E)\text{ for all }E\in\mathcal{B}\},$$

then we can show easily that f_m lies in H and $\int_E f_m\,d\mu=\nu(E)$ for all E contained in $\{f_m<m\}$. Then we can define the function f_0 and the set D in Theorem 2 as in [5]. We feel that it is really a matter of taste whether one prefers this or the approach we have taken. Nevertheless we feel that our approach of finding a maximizer for the inequality $\int_E f\,d\mu\le\nu(E)$ is more natural than finding the maximizer of the quadratic inequality in [5].

References

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