MINKOWSKI'S INTEGRAL INEQUALITY FOR FUNCTION NORMS[1]

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Dedicated to Professor Dr. A.C. Zaanen on the occasion of his eightieth birthday

Let ρ and λ be Banach function norms with the Fatou property. Then the generalized Minkowski integral inequality $\rho(\lambda(f_x)) \leq M\lambda(\rho(f^y))$ holds for all measurable functions $f(x,y)$ and some fixed constant M if and only if there exists $1 \leq p \leq \infty$ such that λ is p–concave and ρ is p–convex.

INTRODUCTION

Let (X,μ) and (Y,ν) be σ-finite measure spaces. Let $0 < r \leq s < \infty$ and let $f(x,y)$ be a $\mu\times\nu$–measurable function. Then the classical integral inequality of Minkowski states that

$$(\int(\int |f(x,y)|^r d\nu(y))^{\frac{s}{r}} d\mu(x))^{\frac{1}{s}} \leq (\int(\int |f(x,y)|^s d\mu(x))^{\frac{r}{s}} d\nu(y))^{\frac{1}{r}}.$$

If we define for fixed $x \in X$ the function f_x by $f_x(y) = f(x,y)$ and for fixed $y \in Y$ the function f^y by $f^y(x) = f(x,y)$, then the above inequality is the same as

$$\|\|f_x\|_r\|_s \leq \|\|f^y\|_s\|_r.$$

The goal of this paper is to extend this inequality to function norms. For general information and terminology concerning function norms and Banach function spaces we refer

[1]This a corrected version of the published version, dated November 13, 2014. The proof of Theorem 2.3 contained an error as pointed out to the author by Rovshan Bandaliyev.

to [10]. It is known that if ρ is a function norm that $\rho(f_x)$ need not be a measurable function on X (see [5]). To avoid this pathology we shall assume that all our function norms have the Fatou property, i.e., if $0 \le f_k \uparrow f$, $f_k \in L_\rho$ and $\sup_k \rho(f_k) < \infty$, then $f \in L_\rho$ and $\rho(f) = \sup_k \rho(f_k)$. A function norm ρ is said to have the weak Fatou property, if $0 \le f_k \uparrow f$, $f_k \in L_\rho$ and $\sup_k \rho(f_k) < \infty$ implies that $f \in L_\rho$. As was shown in [4] the Fatou property is sufficient to ensure the measurability of $\rho(f_x)$, but the weak Fatou property is not (see [5]). We note that the associate norm ρ' of a function norm always has the Fatou property. Let λ be a function norm on $L_0(Y,\nu)$ and ρ a function norm on $L_0(X,\mu)$, where $L_0(Y,\nu)$, respectively $L_0(X,\mu)$, denotes the space of (equivalence classes of) all measurable functions on Y, respectively X. The main result of this paper is that

$$\rho(\lambda(f_x)) \le M\lambda(\rho(f^y)) \tag{*}$$

holds for all measurable functions $f(x,y)$ and some fixed constant M if and only if there exists $1 \le p \le \infty$ such that λ is p–concave and ρ is p–convex. We note that equation (*) generalizes Minkowski's integral inequality to arbitrary function norms. In section 2 of this paper the above mentioned result will be proved. The proof of this theorem depends crucially on the fundamental result of Krivine about the local structure of a Banach lattice ([3], see [8] for an expository account of this theorem). In section 3 we shall relate the main result to the theory of integral operators of finite double norm (the so called Hille-Tamarkin integral operators) and integral operators of finite inverse double norm.

1. PRELIMINARIES

Let us recall the notion of p–convexity and p–concavity. Let ρ be a Banach function norm and let $L_\rho = L_\rho(X,\mu)$ denote the corresponding Banach function space. Then L_ρ is called p–convex for $1 \le p \le \infty$ if there exists a constant M such that for all $f_1, \ldots, f_n \in L_\rho$,

$$\rho((\sum_{k=1}^{n} |f_k|^p)^{\frac{1}{p}}) \le M(\sum_{k=1}^{n} \rho(f_k)^p)^{\frac{1}{p}} \text{ if } 1 \le p < \infty$$

or $\rho(\sup|f_k|) \le M \max_{1\le k\le n} \rho(f_k)$ if $p = \infty$. Similarly L_ρ is called p–concave for $1 \le p \le$

∞ if there exists a constant M such that for all $f_1, \ldots, f_n \in L_\rho$,

$$(\sum_{k=1}^n \rho(f_k)^p)^{\frac{1}{p}} \leq M\rho((\sum_{k=1}^n |f_k|^p)^{\frac{1}{p}}) \text{ if } 1 \leq p < \infty$$

or $\max_{1\leq k\leq n} \rho(f_k) \leq M\rho(\sup |f_k|)$ if $p = \infty$. The notions of p–convexity, respectively p–concavity are closely related to the notions of upper p–estimate (strong ℓ_p–composition property), respectively lower p–estimate (strong ℓ_p–decomposition property) as can be found in e.g. [6, Theorem 1.f.7]. In particular the lower index $s(L_\rho)$ of L_ρ is also equal to the supremum of $\{p \geq 1 : L_\rho \text{ is } p - \text{convex}\}$ and the upper index $\sigma(L_\rho)$ of L_ρ equals the infimum of $\{p \geq 1 : L_\rho \text{ is } p - \text{concave}\}$. We now recall the result of J.L. Krivine (see [8] for a discussion of this result as well as several relevant references).

THEOREM 1.1 (Krivine). *Let E be an infinite dimensional Banach lattice. Then for all integers n, all $\epsilon > 0$, $p = s(E)$ and $p = \sigma(E)$ there exist disjoint $x_1, \ldots, x_n$ in E such that for all n–tuples $\{a_i\}$ of real numbers we have*

$$\|\{a_i\}\|_p \leq \|\sum_{i=1}^n a_i x_i\| \leq (1+\epsilon)\|\{a_i\}\|_p.$$

The following proposition describes p–convex function norms with the weak Fatou property. Its origin is in Pisier's work and the theory of p–concavication (see [6]).

PROPOSITION 1.2. *Let L_ρ be a p–convex Banach function space with the weak Fatou property. Then there exists a collection G of non-negative measurable functions on X such that ρ is equivalent to the function norm*

$$\rho_1(f) = \sup_{g\in G}\left(\int |f|^p g d\mu\right)^{\frac{1}{p}}.$$

PROOF. Define $\tau(f) = (\rho(|f|^{\frac{1}{p}}))^p$. Then τ has the weak Fatou property and all the properties of a norm, except that $\tau(f+g) \leq M(\tau(f) + \tau(g))$, where M is the convexity constant of ρ. Let

$$\tau_1(f) = \inf\{\sum_{i=1}^n \tau(f_i) : |f| = \sum_{i=1}^n |f_i|\}.$$

Then it is easy to see that τ_1 is a function norm with the weak Fatou property, which is equivalent to τ. Let G be the the positive part of the unit ball of L'_{τ_1}, i.e., $G = \{g \geq 0 : \tau'_1(g) \leq 1\}$. It follows from [10, Theorem 112.2] that τ_1 is equivalent to the second associate norm τ''_1, i.e., τ_1 is equivalent to $\sup\{\int |f|gd\mu : g \in G\}$. The conclusion now follows easily, since $\rho(f) = (\tau(|f|^p))^{\frac{1}{p}}$.

We note that the function norm ρ_1 has the Fatou property, so that the assumption that ρ has the weak Fatou property is necessary in the above proposition. If one assumes that the convexity constant M equals one and that ρ has the Fatou property, then $\rho = \rho_1$.

2. MINKOWSKI'S INEQUALITY FOR FUNCTION NORMS

We start with the special case that $\lambda = \|\cdot\|_1$ or $\rho = \|\cdot\|_\infty$.

PROPOSITION 2.1. *Let ρ and λ be function norms with the Fatou property and let $f(x, y)$ be a measurable function. Then we have*

$$\rho\big(\|f_x\|_1\big) \leq \|\rho(f^y)\|_1$$

and

$$\|\lambda(f_x)\|_\infty \leq \lambda\big(\|f^y\|_\infty\big).$$

PROOF. Let $0 \leq g \in L'_\rho$ with $\rho'(g) \leq 1$. Then we have

$$\begin{aligned}\int \Big(\int |f(x,y)|d\nu(y)\Big)g(x)d\mu(x) &= \int \Big(\int |f(x,y)|g(x)d\mu(x)\Big)d\nu(y) \\ &\leq \int \rho(f^y)d\nu(y) = \|\rho(f^y)\|_1.\end{aligned}$$

By taking the supremum over the collection of all such g we obtain

$$\rho\big(\|f_x\|_1\big) \leq \|\rho(f^y)\|_1.$$

This proves the first inequality. The second inequality can be proved along similar lines or by a duality argument.

COROLLARY 2.2 (Minkowski's integral inequality). *Let $0 < r \le s < \infty$ and let $f(x,y)$ be a $\mu \times \nu$–measurable function. Then we have*

$$(\int(\int |f(x,y)|^r d\nu(y))^{\frac{s}{r}} d\mu(x))^{\frac{1}{s}} \le (\int(\int |f(x,y)|^s d\mu(x))^{\frac{r}{s}} d\nu(y))^{\frac{1}{r}}.$$

PROOF. Let $p = \frac{s}{r}$ and take $\rho = \|\cdot\|_p$. Then apply the above proposition to $|f(x,y)|^r$ to get

$$\begin{aligned}(\int(\int |f(x,y)|^r d\nu(y))^{\frac{s}{r}} d\mu(x))^{\frac{r}{s}} &= \rho(\||f_x|^r\|_1) \\ &\le \|\rho(|f^y|^r)\|_1 = (\int |f(x,y)|^s d\mu(x))^{\frac{r}{s}} d\nu(y)).\end{aligned}$$

THEOREM 2.3. *Let ρ and λ be function norms with the Fatou property and assume that there exists $1 \le p \le \infty$ such that ρ is p–convex and λ is p–concave. Then there exists a constant C such that for all measurable $f(x,y)$ we have*

$$\rho(\lambda(f_x)) \le C\lambda(\rho(f^y)).$$

PROOF. If $p = \infty$, then ρ is equivalent to $\|\cdot\|_\infty$ and the theorem follows then from Proposition 2.1. Assume therefore that $1 \le p < \infty$. Then λ is an order continuous norm. We shall first prove the theorem under the additional hypothesis that also ρ is weighted p-seminorm. In that case the product seminorms $\rho\lambda$ and $\lambda\rho$ are order continuous and it therefore suffices to prove the inequality for functions in the collection $\mathcal{P} = \{f(x,y) : f(x,y) = \sum_{i=1}^n f_i(x)g_i(y), f_i, g_i \ge 0, f_i, g_i \text{ measurable}, \{f_i\} \text{ mutually disjoint}\}$. Let $f = \sum_{i=1}^n f_i g_i \in \mathcal{P}$. Let M denote a convexity constant of ρ and a concavity constant of λ. Assume first that the convexity constant of $\rho = 1$. Then by proposition 1.2 we have $\rho(f) = \sup_{g\in G} \|f\|_{p,g}$, where $\|\cdot\|_{p,g}$ denotes the weighted p-seminorm of Proposition 1.2.

Let $\|\cdot\|_{p,g}$ be one of these seminorms. Then we have the following inequalities

$$\begin{aligned}\|(\lambda(f_x))\|_{p,g} &\le \|(\sum_{i=1}^{n} f_i\lambda(g_i))\|_{p,g}\\ &= \big(\sum_{i=1}^{n} \|f_i\|_{p,g}^p \lambda(g_i)^p\big)^{\frac{1}{p}}\\ &= \big(\sum_{i=1}^{n} \lambda(\|f_i\|_{p,g} g_i)^p\big)^{\frac{1}{p}}\\ &\le M\lambda\big(\big(\sum_{i=1}^{n} \|(f_i)\|_{p,g}^p g_i(y)^p\big)^{\frac{1}{p}}\big) = M\lambda(\|f^y\|_{p,g}),\end{aligned}$$

for all $f \in \mathcal{P}$. Hence for all measurable functions $f \ge 0$ we have that

$$\|(\lambda(f_x))\|_{p,g} \le M\lambda(\|f^y\|_{p,g}) \le M\lambda(\rho(f^y)).$$

Taking now on the left the supremum over all $\|\cdot\|_{p,g}$ we conclude that $\rho(\lambda(f_x)) \le M\lambda(\rho(f^y))$ for all measurable $f \ge 0$ in case ρ has convexity constant equal to 1. For the case of constant of convexity equal to M, we pass to an equivalent norm with convexity constant equal to one and apply the above proof. We conclude that $\rho(\lambda(f_x)) \le C\lambda(\rho(f^y))$ for all $f \ge 0$.

REMARK. Note that if $\sigma(L_\lambda) < s(L_\rho)$, then for any $\sigma(L_\lambda) < p < s(L_\rho)$ we have that ρ is p–convex and λ is p–concave.

As a first step in showing that the conditions of the above theorem are also necessary, we prove the following proposition.

PROPOSITION 2.4. *Let ρ and λ be function norms with the Fatou property and assume that there exists a constant C such that for all measurable $f(x,y)$ we have*

$$\rho(\lambda(f_x)) \le M\lambda(\rho(f^y)).$$

Then $\sigma(L_\lambda) \le s(L_\rho)$.

PROOF. Let $p = s(L_\rho), q = \sigma(L_\lambda)$ and let n be a positive integer. Then by Krivine's theorem (Theorem 1.1) there exist disjoint $g_1, \ldots, g_n$ in L_ρ and disjoint $h_1, \ldots, h_n$

in L_λ such that for all n–tuples $\{a_i\}$ of real numbers we have

$$\|\{a_i\}\|_p \le \rho\big(\sum_{i=1}^n a_i g_i\big) \le 2\|\{a_i\}\|_p$$

and

$$\|\{a_i\}\|_q \le \lambda\big(\sum_{i=1}^n a_i h_i\big) \le 2\|\{a_i\}\|_q.$$

Let $f(x,y) = \sum_{i=1}^n a_i g_i(x) h_i(y)$. Then we have

$$\begin{aligned}\lambda(\rho(f^y)) &= \lambda\big(\sum_{i=1}^n |a_i| \rho(g_i) h_i(y)\big) \\ &\le 2\lambda\big(\sum_{i=1}^n |a_i| h_i\big) \le 4\|\{a_i\}\|_q\end{aligned}$$

and

$$\begin{aligned}\rho(\lambda(f_x)) &= \rho\big(\sum_{i=1}^n |a_i| g_i(x) \lambda(h_i)\big) \\ &\ge \rho\big(\sum_{i=1}^n |a_i| g_i\big) \ge \|\{a_i\}\|_p.\end{aligned}$$

From this it follows that $\|\{a_i\}\|_p \le 4C\|\{a_i\}\|_q$ for all n–tuples $\{a_i\}$, where C is independent of n. This implies that $q \le p$, which concludes the proof of the proposition.

We now prove the converse of Theorem 2.3.

THEOREM 2.5. *Let ρ and λ be function norms with the Fatou property and assume that there exists a constant C such that for all measurable $f(x,y)$ we have*

$$\rho(\lambda(f_x)) \le M\lambda(\rho(f^y)).$$

Then there exists $1 \le p \le \infty$ such that ρ is p–convex and λ is p–concave.

PROOF. From the above proposition it follows that $\sigma(L_\lambda) \le s(L_\rho)$. In case we have that $\sigma(L_\lambda) < s(L_\rho)$, then the theorem will hold for any p such that $\sigma(L_\lambda) < p < s(L_\rho)$. Hence assume that $\sigma(L_\lambda) = s(L_\rho) = p$. By Krivine's theorem (Theorem 1.1) there exist disjoint $h_1, \ldots, h_n$ in L_λ such that for all n–tuples $\{a_i\}$ of real numbers we have

$$\|\{a_i\}\|_p \le \lambda\big(\sum_{i=1}^n a_i h_i\big) \le 2\|\{a_i\}\|_p.$$

Let $g_1, \ldots, g_n$ be in L_ρ and put $f(x,y) = \sum_{i=1}^n g_i(x)h_i(y)$. Then we have

$$\lambda(\rho(f^y)) = \lambda\big(\sum_{i=1}^n \rho(g_i)h_i\big) \leq 2\|\{\rho(g_i)\}\|_p$$

and

$$\rho(\lambda(f_x)) \geq \rho\big(\|\{g_i\}\|_p\big).$$

From $\rho(\lambda(f_x)) \leq M\lambda(\rho(f^y))$ it follows now that

$$\rho\big(\big(\sum_{i=1}^n |g_i|^p\big)^{\frac{1}{p}}\big) \leq 2M\big(\sum_{i=1}^n \rho(g_i)^p\big)^{\frac{1}{p}},$$

i.e., ρ is p–convex. By a similar argument (or by duality) one can show that λ is p–concave.

The following corollary is due to A.V. Buhvalov ([1]), the special case $\rho = \lambda$ is due to N.J. Nielsen ([7]).

COROLLARY 2.6(Generalized Kolmogorov-Nagumo's Theorem).*Let ρ and λ be function norms with the Fatou property and assume that the double norm $\rho(\lambda(f_x))$ is equivalent with the double norm $\lambda(\rho(f^y))$. Then there exists $1 \leq p \leq \infty$ such that ρ and λ are equivalent to an L_p–norm.*

PROOF. Applying the above theorem twice we get that there exist $1 \leq p, q \leq \infty$ such that ρ is p–convex and q–concave, and λ is q–convex and p–convex. Hence it follows that $p = q$. If $p < \infty$ the result now follows from [6, Lemma 1.b.13]. In case $p = \infty$ an inspection of the proof of [6, Lemma 1.b.13] shows that ρ and λ are equivalent to an AM-norm.

3. INTEGRAL OPERATORS OF FINITE DOUBLE NORM

Recall that a linear operator T from L_λ into L_ρ is called an integral operator if there exists a $\mu \times \nu$–measurable function $T(x,y)$ on $X \times Y$ such that

$$\int |T(x,y)f(y)|d\nu(y) < \infty$$

a.e. for all $f \in L_\lambda$ and such that

$$Tf(x) = \int T(x,y)f(y)d\nu(y)$$

a.e. for all $f \in L_\lambda$. Such an integral operator is called an integral operator of finite double norm, or Hille-Tamarkin operator, if $\rho(\lambda'(T_x)) < \infty$. Characterizations and compactness properties of such operators are discussed in [9] (see also the references in [9]). An integral operator is called an integral operator of finite inverse double norm if $\lambda'(\rho(T^y)) < \infty$. Integral operators which are of finite double and finite inverse double norm are sometimes called integral operators of complete finite double norm. It is well known that the spaces of integral operators of finite double norm and of finite inverse double norm are Banach function spaces with respect to the product norm $\rho(\lambda'(\cdot))$, respectively $\lambda'(\rho(\cdot))$. Our main results of the previous section imply the following theorem.

THEOREM 3.1. *Let λ and ρ be Banach function norms with the Fatou property. Then the following holds.*

(1) *Every integral operator of finite inverse double norm is an integral operator of finite double norm if and only if there exists $1 \leq p \leq \infty$ such that ρ is p–convex and λ is p'–convex, where $\frac{1}{p} + \frac{1}{p'} = 1$.*
(2) *Every integral operator of finite double norm is an integral operator of finite inverse double norm if and only if there exists $1 \leq p \leq \infty$ such that ρ is p'–concave and λ is p–concave, where $\frac{1}{p} + \frac{1}{p'} = 1$.*

PROOF. Every integral operator of finite inverse double norm is an integral operator of finite double norm if and only if there exists a constant M such that $\rho(\lambda'(T_x)) \leq M\lambda'(\rho(T^y))$. From Theorem 2.3 and Theorem 2.5. it follows that this inequality holds if and only if there exists $1 \leq p \leq \infty$ such that ρ is p–convex and λ' is p–concave. Now λ' is p–concave implies that $\lambda'' = \lambda$ is p'–convex. This proves (1). The proof of (2) is similar and therefore omitted.

Integral operators of complete finite double norm occur naturally in the study of power summability of eigenvalues of integral operators. In [2] it was e.g. proved that on an order continuous Banach function space the eigenvalues of an integral operator of complete finite double norm are always 4th-power summable. More precise exponents of summability were then given in terms of the lower and upper indices of the Banach function space. The following corollary of Theorem 3.1 shows that in some cases integral operators of finite (inverse) double norm are of complete finite double norm.

COROLLARY 3.2. *Let $X = Y$ and let $\rho = \lambda$ be a Banach function norm with the Fatou property. Then the following holds.*

(1) *Every integral operator of finite inverse double norm from L_ρ into L_ρ is an integral operator of finite double norm if and only if there exists $2 \leq p \leq \infty$ such that ρ is p–convex.*

(2) *Every integral operator of finite double norm from L_ρ into L_ρ is an integral operator of finite inverse double norm if and only if there exists $1 \leq p \leq 2$ such that ρ is p–concave.*

(3) *The collection of integral operators of finite double norm from L_ρ into L_ρ coincides with the collection of integral operators of inverse finite double norm from L_ρ into L_ρ if and only if L_ρ is lattice isomorphic to an L_2–space.*

PROOF. By the above theorem every integral operator of finite inverse double norm from L_ρ into L_ρ is an integral operator of finite double norm if and only if there exists $1 \leq p \leq \infty$ such that ρ is p–convex and ρ is p'–convex. Since at least one of p and p' is greater than or equal to 2, part (1) follows. Part (2) follows similarly. To prove part (3) note first that when L_ρ is lattice isomorphic to an L_2–space then the collections of integral operators of finite double norm, respectively finite inverse double norm both coincide with the collection of Hilbert-Schmidt operators. Now by parts (1) and (2) The collection of integral operators of finite double norm from L_ρ into L_ρ coincides with the collection of integral operators of inverse finite double norm from L_ρ into L_ρ if and only if L_ρ is both 2–convex and 2–concave and the result follows from this as in Corollary 2.6.

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