Homework 9, due April 4

1. **a.** Let $f : [a,b] \to \mathbb{R}$ be differentiable at the point $x \in (a,b)$. Prove that

$$\lim_{h \downarrow 0} \frac{f(x+h) - f(x-h)}{2h} = f'(x).$$

b. Prove Lebesgue's density theorem: Let $E \subset [a,b]$ be a measurable set. Then

$$\lim_{h \downarrow 0} \frac{1}{2h} m(E \cap [x-h, x+h]) = \begin{cases} 1 & \text{for a.e. } x \in E \\ 0 & \text{for a.e. } x \notin E \end{cases}$$

2. Let $f : [a,b] \to \mathbb{R}$ be an increasing function such that

$$\int_a^b f'(t)\, dt = f(b) - f(a).$$

Prove that f is absolutely continuous.

3. Let $f : [a,b] \to \mathbb{R}$ be a function which satisfies the definition of absolutely continuity without requiring the intervals to be disjoint. Show that f satisfies a Lipschitz condition. Hint: Let $\delta > 0$ correspond to $\epsilon = 1$. The claim is that than $|f(x) - f(y)| \leq \frac{2}{\delta}|x-y|$. If not, there exists $x < y$ such that $|f(x) - f(y)| > \frac{2}{\delta}|x-y|$. Choose now m, n such that $\frac{2}{\delta} < \frac{m}{n}|x-y| < \delta$ and partition $[x,y]$ as $x = x_0 < \cdots < x_n = y$ such that $|x_i - x_{i-1}| = |x-y|/n$. Now use the hypothesis on the single interval (x_{i-1}, x_i) used m-times to see that $m|f(x_i) - f(x_{i-1})| < 1$. Now sum $m|f(x_i) - f(x_{i-1})|$ from 1 to n and bound this from below by n and from above by n to get a contradiction.

4. Prove the Ratio test: Let $\sum_{n=0}^{\infty} c_n (z-a)^n$ be a power series with radius of convergence R, i.e., $\frac{1}{R} = \overline{\lim}\, |c_n|^{\frac{1}{n}}$. Assume $c_n \neq 0$ for all n. Then

$$\underline{\lim} \left| \frac{c_{n+1}}{c_n} \right| \leq \frac{1}{R} \leq \overline{\lim} \left| \frac{c_{n+1}}{c_n} \right|.$$

In particular, if $\lim_{n \to \infty} \left| \frac{c_{n+1}}{c_n} \right|$ exists, then $\frac{1}{R} = \lim_{n \to \infty} \left| \frac{c_{n+1}}{c_n} \right|$. Hint: Putting $a_n = |c_n|$, prove that

$$\underline{\lim} \frac{a_{n+1}}{a_n} \leq \underline{\lim} \sqrt[n]{a_n} \leq \overline{\lim} \sqrt[n]{a_n} \leq \overline{\lim} \frac{a_{n+1}}{a_n}.$$