Homework 9

(1) Let $f \in L^p$ for some $1 \leq p < \infty$ and let $f_h(x) = f(x-h)$. Prove that $\|f - f_h\|_p \to 0$ as $h \to 0$. Give an example to show that the statement is false for $p = \infty$. (Hint: the case $p = 1$ is on pg 74 of the text.)

(2) Let $f \in L^p$ with $1 \leq p < \infty$ and $g \in L^q$, where $\frac{1}{p} + \frac{1}{q} = 1$.
 a. Prove that $f * g \in L^\infty$.
 b. Prove that $f * g$ is uniformly continuous. (Hint: use problem 1.)

(3) Let $E \subset [0,1] \times [0,1]$ be a measurable set. Assume that $m(E_x) \leq \frac{1}{2}$ for almost every $x \in [0,1]$. Prove that $m(\{y \in [0,1] : m(E^y) = 1\}) \leq \frac{1}{2}$.

(4) Let $f \in L^1(\mathbb{R})$ and define for $h > 0$

$$\phi_h(x) = \frac{1}{2h} \int_{x-h}^{x+h} f(t)\, dt.$$

Prove that ϕ_h is integrable and $\|\phi_h\|_1 \leq \|f\|_1$.