Homework 9.

(1) Let $f_n(x) = \frac{x^{2n}}{1+x^{2n}}$. Prove that $f(x) = \lim_{n\to\infty} f_n(x)$ exists for all $x \in \mathbb{R}$. Does (f_n) converge uniformly to f?

(2) Define $f_n : [0,1] \to [0,1]$ by $f_n(x) = x^n(1-x)$. Prove that f_n converges uniformly to 0.

(3) Prove that
$$f_n(x) = \frac{nx + \sin(nx^2)}{n}$$
converges uniformly to f on $[0,1]$, where $f(x) = x$.

(4) Let $f_n(x) = x^n e^{-nx}$. Prove that $\sum f_n$ converges uniformly (Hint: Use the Weierstrass M-test).

(5) Prove that $\sum_{n=1}^{\infty} \frac{nx^2}{n^3+x^3}$ converges uniformly on $[0,2]$.