Homework 8, Additional Problems.

(1) **a.** Let $f \in L^r \cap L^\infty$ for some $r < \infty$. Prove that

$$\|f\|_p \leq \|f\|_r^{\frac{r}{p}} \|f\|_\infty^{1-\frac{r}{p}}$$

for all $r < p < \infty$.

b. Asume $f \in L^r \cap L^\infty$ for some $r < \infty$. Prove

$$\lim_{p\to\infty} \|f\|_p = \|f\|_\infty.$$

(Hint: Use **a.**to get an upper bound for $\overline{\lim}_{p\to\infty} \|f\|_p$ and then use that for $0 \leq t < \|f\|_\infty$ the set $A = \{x : |f(x)| \geq t\}$ has positive measure)

(2) Let $f_n, f \in L^p$ with $1 \leq p < \infty$. Assume $f_n(x) \to f(x)$ a.e. Prove $f_n \to f$ in L^p if and only if $\|f_n\|_p \to \|f\|_p$.

(3) Let $f \in L^1$. Denote by f_h the function $f_h(x) = f(x - h)$. Prove that $\|f - f_h\|_1 \to 0$ as $h \to 0$. (Hint: Prove this first for f a continuous function with compact support.)