Extra problems Homework 7.

(1) Let $f : [0, 1 \to \mathbb{R}$ be a continuous function. Prove that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f(k/n) = \int_0^1 f(x)\, dx.$$

(Hint: Use uniform continuity to show that for $\epsilon > 0$ there exists a $\delta > 0$ such that $S(\sigma) - s(\sigma) < \delta$ for all subdivisions σ with norm $N(\sigma) < \delta$.)

(2) Use the previous problem (by picking the right function f) to evaluate the following sums, where you can use calculus to evaluate the integral on the right.

(a)

$$\lim_{n\to\infty} \frac{1}{n^3} \sum_{k=1}^{n} k^2$$

(b)

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{n}{n^2 + k^2}.$$