Homework 7.

(1) Let $c_n > 0$ in $\mathbb{R}$. Prove that

$$\liminf \frac{c_{n+1}}{c_n} \leq \liminf \sqrt[n]{c_n} \leq \limsup \sqrt[n]{c_n} \leq \limsup \frac{c_{n+1}}{c_n}.$$

In particular, if $\lim_{n\to\infty} \frac{c_{n+1}}{c_n}$ exists, then $\lim_{n\to\infty} \sqrt[n]{c_n} = \lim_{n\to\infty} \frac{c_{n+1}}{c_n}$.

(2) Let $a_n \geq 0$ and $b_n \geq 0$. Assume that both (a_n) and (b_n) are bounded sequences.
   (a) Prove that $\limsup a_n b_n \leq (\limsup a_n)(\limsup b_n)$.
   (b) Give an example that we can have strict inequality in (a).
   (c) Prove $\limsup a_n^2 = (\limsup a_n)^2$.