Homework 7, due March17

1. **a.** For $m(E) < \infty$, show that

$$L^\infty(E) \subset L^r(E) \subset L^p(E) \subset L^1(E),$$

where $1 < p < r < \infty$. Show, for $E = (0, 1]$, by example that all the inclusions can be strict.

b. Show that in general (i.e., if $m(E) = \infty$)

$$L^\infty \cap L^1 \subset L^p \subset L^\infty + L^1 = \{f : f = g + h, g \in L^\infty, h \in L^1\}.$$

2. Let $f \in L^2([0, 1])$. Prove that

$$(\int_{[0,1]} xf(x)\,dx)^2 \le \frac{1}{3}\int_{[0,1]} |f(x)|^2\,dx.$$

3. Let E be a measurable set of finite measure and let $1 < p < \infty$. Assume $f_n \in L^p(E, m)$ such that $\|f_n\|_p \le 1$ and $f_n(x) \to 0$ a.e. Prove that $\|f_n\|_1 \to 0$.

4. Let $f_n \to f$ in L^p, $1 \le p < \infty$, and let $\{g_n\}$ be a sequence of measurable functions such that $|g_n| \le M$ for all n, and $g_n \to g$ a.e.

a. Prove $\|(g_n - g)f\|_p \to 0$.

b. Prove $g_n f_n \to fg$ in L^p.