Homework 7.

(1) Let f be continuous on $[0,1]$. Prove that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f\left(\frac{k}{n}\right) = \int_0^1 f(x)\,dx.$$

(Hint: Look at the proof that continuous functions are Riemann integrable.)

(2) Use problem 1 to compute the following limit (you can use calculus to evaluate the definite integral)

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{k}{n^2+k^2}.$$

(3) Let $f, g, h : [a,b] \to \mathbb{R}$ be continuous and let $g, h : (a,b) \to (a,b)$ be differentiable with $g(x) \geq h(x)$ for all x. Define$F(x) = \int_{h(x)}^{g(x)} f(t)\,dt$. Find $F'(x)$. Hint: do first the case that one of g or h is constant.

(4) Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. For $a > 0$ define $g(x) = \int_{x-a}^{x+a} f(t)\,dt$. Prove that g is differentiable and find $g'(x)$.

(5) Let $f : [0,1] \to \mathbb{R}$ be continuous Prove that

$$\int_0^1 f(x^n)\,dx \to f(0),$$

as $n \to \infty$. Hint: Find first $b < 1$ so that $\int_b^1 |f(x^n)|\,dx + |f(0)|(1-b) < \frac{\epsilon}{2}$. Then use that the sequence $f_n(x) = f(x^n)$ converges uniformly to $f(0)$ on $[0,b]$.