Homework 6.

(1) Let $f : [0, 1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1 & \text{if } x \in \{\frac{1}{n} : n \in \mathbb{N}\} \\ 0 & \text{for all } x \neq \frac{1}{n}. \end{cases}$$

**a.** Prove that $f$ is Riemann integrable on $[c, 1]$ for all $0 < c < 1$ and that $\int_c^1 f(x)\,dx = 0$.

**b.** Prove that $f$ is Riemann integrable on $[0, 1]$ and that $\int_0^1 f(x)\,dx = 0$.

(2) Let $f : [a, b] \to \mathbb{R}$ be Riemann integrable.

**a.** Prove that if $f$ is continuous at $c \in [a, b]$ and $f(c) \neq 0$, then

$$\int_a^b |f(x)|\,dx > 0.$$

**b.** Prove that if $f$is continuous on $[a, b]$, then $\int_a^b |f(x)|\,dx = 0$ if and only if $f(x) = 0$ for all $x \in [a, b]$.

**c.** Does **b.** hold if the absolute value is removed? Prove or give a counterexample.

(3) Let $f : [a, b] \to \mathbb{R}$ be continuous such that $\int_a^c f(x)\,dx = 0$ for all $a < c \leq b$. Prove that $f(x) = 0$ for all $x \in [a, b]$.

(4) Let $f : [0, 1] \to \mathbb{R}$ be Riemann integrable. Prove that

$$\lim_{n\to\infty} \int_0^1 x^n f(x)\,dx = 0.$$

(5) Let $f : [a, b] \to \mathbb{R}$ be continuous and set $M = \max\{|f(x)| : x \in [a, b].\}$. Assume $M \neq 0$.

**a.** Prove that for every $\epsilon > 0$ there exists an interval $[c, d] \subset [a, b]$ such that

$$(M - \epsilon)^n (d - c) \leq \int_a^b |f(x)|^n\,dx \leq M^n (b - a).$$

**b.** Prove that

$$\lim_{n\to\infty} \left( \int_a^b |f(x)|^n\,dx \right)^{\frac{1}{n}} = M.$$