Homework 5 additional problems.

(1) Let f, g be bounded uniformly continuous functions from $\mathbb{R}$ to $\mathbb{R}$.
  a. Prove that the product fg is again uniformly continuous on $\mathbb{R}$.
  b. Show by means of an example that part a) fails if we omit the boundedness assumption.

(2) Let $f; (a, b) \to \mathbb{R}$ be uniformly continuous.
  a. Prove that f maps Cauchy sequences to Cauchy sequences.
  b. Prove that $\lim_{x \downarrow a} f(x)$ and $\lim_{x \uparrow b} f(x)$ exist.
  c. Prove that f has a continuous extension to $[a, b]$.

(3) Show that

$$\{(x, y) : x = 0, -1 \leq y \leq 1\} \cup \{(x, y) : 0 < x \leq 1, y = \sin \frac{1}{x}\}$$

is a connected subset of $\mathbb{R}^2$.