Homework 4, Additional Problems.

(1) Let $f : E \to \mathbb{R} \cup \{\pm\infty\}$ be measurable. Prove that $f^{-1}(B)$ is measurable for all Borel sets $B \subset \mathbb{R}$.

(2) Let F be the Cantor-Lebesgue function from $[0,1] \to [0,1]$ and define $F_1(x) = x + F(x)$.

a. Show that F_1 is one-to-one and onto $[0,2]$ and that its inverse is continuous (Hint: Use that F_1 is strictly increasing).

b. Show that $F_1(\mathcal{C})$ has measure one.

c. Let G be the function F_1^{-1}. Show that there is a measurable set E such that $G^{-1}(E)$ is not measurable. As a consequence show that this set E can't be a Borel set.

(3) **a.** Let $A \subset \mathcal{R}^d$. Prove there exists a measurable set $E \supset A$ such that $m^*(A) = m(E)$.

b. Let $A_1 \subset A_2 \subset \cdots \uparrow A$. Prove that $m^*(A_n) \uparrow m^*(A)$.