Homework 3, Additional Problems.

(1) Let $(X, d)$ be a metric space.

**a.** Let $E_i \subset X$ $(i \in \{1, \cdots, n\})$ be a finite collection of subsets of X. Prove that

$$\overline{\cup_{i=1}^n E_i} = \cup_{i=1}^n \overline{E_i}.$$

**b.** Let $E_i (i \in I)$ now be an arbitrary collection of subsets of $X$. Prove that

$$\overline{\cap_{i \in I} E_i} \subset \cap_{i \in I} \overline{E_i}$$

and give an example that in general (even for two sets) that the inclusion is proper.

(2) Let $(X, d)$ be a metric space and let $A \subset X$ be a non-empty subset. Define

$$d(x, A) = \inf\{d(x, y) : y \in A\}.$$

**a.** Prove $d(x, A) = 0$ if and only if $x \in \overline{A}$.

**b.** Show that

$$|d(x, A) - d(y, A)| \leq d(x, y),$$

for all $x, y \in X$.

**c.** Let now $A, B \subset X$ be disjoint closed subsets. Prove there exists a continuous $f : X \to [0, 1]$ such that $f(x) = 0$ for all $x \in A$ and $f(x) = 1$ for all $x \in B$. (Hint: Show that $f(x) = \frac{d(x,A)}{d(x,A)+d(x,B)}$ works.)