Homework 3, due February 5

1. Let $(X, \mathcal{B}, \mu)$ be a measure space and let $f \in L_p$ with $1 \leq p < \infty$.

a. Prove that $\{x : |f(x)| > 0\}$ has σ–finite measure (i.e., is a countable union of sets of finite measure).

b. Prove that for all $\epsilon > 0$ there exists a set E_ϵ of finite measure such that

$$\int_{\tilde{E}_\epsilon} |f|^p \, d\mu < \epsilon.$$

2. Let $f \in L_2([0,1], m)$. Prove that

$$\left(\int_{[0,1]} x f(x) \, dx\right)^2 \leq \frac{1}{3} \int_{[0,1]} |f(x)|^2 \, dx.$$

(Here m and dx both refer to Lebesgue measure on $[0,1]$.)

3. Let $(X, \mathcal{B}, \mu)$ be a finite measure space and let $1 < p < \infty$. Assume $f_n \in L_p(X, \mu)$ such that $\|f_n\|_p \leq 1$ and $f_n(x) \to 0$ a.e. Prove that $\|f_n\|_1 \to 0$.

4. Let $(X, \mathcal{B}, \mu)$ be a measure space. Let g_n be measurable functions such that $\|g_n\|_\infty \leq 1$ for all n and $\int_E g_n \, d\mu \to 0$ for all $E \in \mathcal{B}$ with $\mu(E) < \infty$. Prove that $\int f g_n \, d\mu \to 0$ for all $f \in L_1(X, \mu)$.