Homework 1, Additional Problem.

(1) Let $1 < p < \infty$ a real number and let q be defined by $1 = \frac{1}{p} + \frac{1}{q}$.

a. Let $f(t) = \frac{1}{p}t^p + \frac{1}{q} - t$. Show (by means of calculus), that $f(t) \geq 0$ for all $t \geq 0$.

b. Show that $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$ for all $a, b > 0$. (Hint: Take $t = \frac{a}{b^{q-1}}$ in part **a.**).

c. Show that $|\sum_{i=1}^n a_i b_i| \leq (\sum_{i=1}^n |a_i|^p)^{\frac{1}{p}} (\sum_{i=1}^n |b_i|^q)^{\frac{1}{q}}$ for all $a = (a_1, \cdots, a_n), b = (b_1, \cdots, b_n) \in \mathbb{R}^n$.

d. Show that

$$\left(\sum_{i=1}^n |a_i + b_i|^p\right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^n |a_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^n |b_i|^p\right)^{\frac{1}{p}}.$$

(Hint: Bound on the left $|a_i + b_i|^p$ by $|a_i||a_i + b_i|^{p-1} + |b_i||a_i + b_i|^{p-1}$ and apply part **c.** to each of the two sums.)

e. Show that $d(x, y) = (\sum_{i=1}^n |x_i - y_i|^p)^{\frac{1}{p}}$ is a metric on $\mathbb{R}^n$.